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F. GUTEKUNST, PHILA.

On Primitive Groups of Odd Order.

By Henry Lewis Rietz.

Introduction.

In his "Theory of Groups of Finite Order" (1897), p. 379, Burnside has called attention to the fact that no simple group of odd composite order is known to exist. Several articles* have recently appeared bearing on this question, in which, among other things, it was proved that no such group can be represented as a substitution group whose degree does not exceed 100. This result was obtained by showing that there is no simple primitive group of odd composite order whose degree falls within the given limits. Burnside determined all the primitive groups of odd order of degree less than 100.†

Since any primitive group of odd order is simply transitive, a study of simply transitive primitive groups may throw light on the question of simple groups of odd order. Some important properties of simply transitive primitive groups have been given by Jordan, Miller, and Burnside.‡

The main objects of the present paper are; first, to make a further study of primitive groups with special reference to those of odd order; secondly, to extend the determination of the primitive groups of odd order to all degrees less than 243.

It results that all groups arrived at in this determination are solvable. From this result it is evident that no simple group of odd composite order can occur

* Miller, Proc. Lond. Math. Soc., Vol. 33, pp. 6–10. Burnside, Proc. Lond. Math. Soc., Vol. 33, pp. 162–185; 257–268. Frobenius, Berliner Sitzungsberichte (1901), pp. 849–858; 1216–1230.

† At the time of the publication of this work, I had also made this determination with the same results.

‡ Jordan, "Traité des Substitutions," pp. 281–284. Miller, Proc. Lond. Math. Soc., Vol. 28, pp. 533–542. Burnside, loc. cit., pp. 162–185.

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as a substitution group of degree less than 243, since if a simple group is represented as a substitution group on the minimum number of letters, it is primitive.

Part I contains a number of theorems, most of which apply to primitive groups whether the order is even or odd, but some use can be made of nearly all of them in determining all the primitive groups of odd order of a given degree. Part II contains the determination of the primitive groups of odd order whose degrees lie between 100 and 243.

I desire to acknowledge my indebtedness to Professor G. A. Miller for helpful suggestions and criticisms during the preparation of this paper.

PART I.

§1.—*On the Number of Substitutions of Degree less than n contained in any Transitive Group of Degree n.*

Let G be any primitive group of composite order g on the elements $a_1, a_2, \ldots, a_n$, and G_s the subgroup leaving a given letter a_s fixed. If $n - \lambda_\alpha$ is the degree of any substitution of G_s, and μ_α the number of substitutions of G_s of this degree, then the total number of substitutions of degree less than n contained in G is

$$\frac{n\mu_1}{\lambda_1} + \frac{n\mu_2}{\lambda_2} + \frac{n\mu_3}{\lambda_3} + \ldots + \frac{n\mu_\rho}{\lambda_\rho}\text{*} \quad \text{or} \quad n\left(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} + \ldots + \frac{\mu_\rho}{\lambda_\rho}\right),$$

where ρ is the number of different degrees occurring among the substitutions of G_s. Since $\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_\rho = \frac{g}{n}$, the above summation in the parentheses may be considered as the sum of just $\frac{g}{n}$ terms of the form $\frac{1}{\lambda}$. We may then rewrite the above expression for the number of substitutions of degree less than n in the form

$$n \sum_{\alpha=1}^{\alpha=\frac{g}{n}} \frac{1}{\lambda_\alpha}. \tag{1}$$

Let x denote the number of systems of intransitivity of G_s, and look upon G_s as having just one system of intransitivity when it is transitive. Then $x + 1$

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* Jordan, Liouville's Journal, Vol. 17 (1872), p. 352.

is the average value of λ_a since the average number of letters in the substitutions of an intransitive group is equal to the excess of the degree over the number of systems of intransitivity.* Hence we have

$$x + 1 = \frac{\sum_{a=1}^{a=\frac{g}{n}} \lambda_a}{\frac{g}{n}}. \tag{2}$$

The λ's in this summation cannot all be equal, since identity is included among the substitutions of G_s. Since the arithmetic mean of any number of positive quantities which are not all equal is greater than their geometric mean, it follows that

$$\frac{\sum_{a=1}^{a=\frac{g}{n}} \lambda_a}{\frac{g}{n}} > \sqrt[\frac{g}{n}]{\lambda_1 \lambda_2 \cdots \cdots \lambda_{\frac{g}{n}}} \tag{3}$$

and

$$\frac{\sum_{a=1}^{a=\frac{g}{n}} \frac{1}{\lambda_a}}{\frac{g}{n}} > \sqrt[\frac{g}{n}]{\frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} \cdots \cdots \frac{1}{\lambda_{\frac{g}{n}}}}. \tag{4}$$

From (3) and (4) it follows that

$$\frac{\left(\frac{g}{n}\right)^2}{\sum_{a=1}^{a=\frac{g}{n}} \lambda_a} < \sum_{a=1}^{a=\frac{g}{n}} \frac{1}{\lambda_a}. \tag{5}$$

From (2) and (5) it follows that

$$\frac{g}{n(x+1)} < \sum_{a=1}^{a=\frac{g}{n}} \frac{1}{\lambda_a} \quad \text{or} \quad \frac{g}{x+1} < n \sum_{a=1}^{a=\frac{g}{n}} \frac{1}{\lambda_a}. \tag{6}$$

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* Jordan, Comptes Rendus, Vol. 74 (1872), p. 977. Frobenius, Crelle's Journal, Vol. 101 (1887), p. 288.

From (1) and (6) we obtain

THEOREM 1.—*In any primitive group G of degree n of composite order g there are more than* $\frac{g}{x+1}$ *substitutions of degree less than n, where x is the number of systems of intransitivity of the subgroup which leaves a given letter fixed.*

In particular, for a multiply transitive group, $x = 1$. Hence,

Cor. 1. *In a multiply transitive group of degree n more than one-half of the substitutions are of degree less than n.*

Cor. 2. *If G is of degree kp (p a prime) and of order mp (m prime to p and $p-1$), the subgroup G_s has at least $p+1$ transitive constituents.*

For a group of this order contains exactly m operators whose orders divide m.* But all the substitutions of degree less than kp would be of orders prime to p. Hence from the above theorem we have $x > \frac{mp}{x+1}$ or $x > p-1$.

Since mp must clearly be an odd number, x must be even. Hence, $x \geqq p+1$.

While it is not our object to treat imprimitive groups, the above theorem can at once be extended to any non-regular transitive group. The only change in the argument is the substitution of $x+m$ in expression (1) for $x+1$, where m represents the number of letters of the transitive group left fixed by the subgroup which leaves a given letter fixed. Hence,

THEOREM 2.—*In any non-regular transitive group of degree n of order g there are more than* $\frac{g}{x+m}$ *substitutions of degree less than n, where x and m are defined as above.*

When applied to known groups, I find that in many cases this simple formula gives very nearly the actual number of substitutions of degree less than the degree of the group.

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* Frobenius, Berliner Sitzungsberichte (1895), p. 1035.

§2.—*Restrictions on the Order of G, when G_s has a Transitive Constituent of Degree p, p^a, pm or pq (p and q primes and $m < p$).*

If G is simply transitive, G_s is intransitive and conversely. Use will frequently be made of the following known theorems:

1. If G_s contains an invariant subgroup H_s of degree $n-\alpha$, H_s is intransitive, and of the n conjugates to which it belongs under G just $\alpha-1$, besides H_s ($H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$) occur in G_s. These $\alpha-1$ subgroups generate a group of degree $n-1$. Furthermore, G_s transforms $H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$ in the same manner as the elements of one of its constituent groups are permuted.*

2. Every prime which divides the order of one transitive constituent of G_s divides the order of each of its constituents.†

THEOREM 3.—*If in G_s all the transitive constituents T_1, T_2, T_3, ... of a given degree t are of orders $s_1, s_2, s_3, \ldots$, and if $\frac{s_1}{t}, \frac{s_2}{t}, \frac{s_3}{t}, \ldots$ do not contain a given prime p occurring as a factor in t, the order of G_s is of the form tk, where k is prime to p.*

We shall assume that the invariant subgroup H_s of G_s corresponding to identity in T_1 is of order $h=\lambda p^m$ (λ prime to p, $m>0$); this must be the case if the theorem is not true. It will be shown that this hypothesis leads to a contradiction. In H_s all the substitutions whose orders are powers of p would generate a group H'_s of order $\lambda' p^m$ invariant in G_s. In the conjugate G_r of G_s, leaving fixed an element of T_1, there occurs just $1/t$ of the substitutions of G_s. Hence the subgroup H'_s would be one of a set of t conjugates transformed by G_r according to one of its transitive constituents T of degree t. In the invariant subgroup H'_r of G_r corresponding to identity in T, all the substitutions are common to G_r and G_s, since they transform H'_s into itself. Now H'_r would be of order $\lambda'' p^m$ (λ'' prime to p). Since in T_1 all the substitutions whose orders are not prime to p are of degree t, all the substitutions whose orders are powers of p common to G_r and G_s are contained in H_s.

If H'_r contained all the substitutions whose orders are powers of p which occur in H_s, the subgroup H'_s would be invariant in G_s and G_r. But this is impossible, since these subgroups are maximal. If H'_r contains only part of these substitutions, let P be such a substitution not contained in H'_r. The order of

* Miller, loc. cit., pp. 534, 535.

† Jordan, loc. cit., p. 284.

$\{H'_r, P\}$ would then be divisible by p^{m+1} and there would be common to G_s and G_r subgroups of order p^{m+1}, which is impossible, since, by hypothesis, the order of H_s is not divisible by p^{m+1}. Hence the theorem.

Cor. 1. *If G_s has a transitive constituent of prime degree p, the order of G_s is not divisible by p^2.*

Cor. 2. *If any number of the transitive constituents of H_s are of a given prime degree p, the constituent group formed of all these transitive constituents is formed by establishing a simple isomorphism between them.*

Cor. 3. *If in G_8 all the transitive constituents of a given degree p^a are of class $p^a - 1$, the order of G_8 is not divisible by p^{a+1}.*

Cor. 4. *If in G_s all the transitive constituents of degree $mp\,(p > m)$ have p systems of imprimitivity, the order of G_8 is not divisible by p^2.*

Lemma. When p and q are distinct primes each of the form $2^m + 1$, there is no imprimitive group of degree pq of odd order whose order is divisible by both p^2 and q^2; and there is no primitive group of degree pq involving in its order only the primes p and q.

The part of this lemma which relates to the imprimitive groups follows at once from the fact, that the only transitive groups of degrees p and q whose orders are odd are the cyclical groups of orders p and q. Suppose there is a primitive group of degree pq of order $p^{a_1}q^{a_2}$. The maximal subgroup G_1, leaving a given letter fixed, is then of degree $pq-1$ and of order $p^{a_1-1}q^{a_2-1}$. Take $p > q$, then, since $p^2 > pq - 1$, no transitive constituent can be of degree $p^\gamma\,(\gamma > 1)$. The transitive constituents cannot all be of degree p, since p is not a divisor of $pq - 1$. Since $pq - 1$ is not divisible by q, we may assume that some of the transitive constituents are of degree p while others are of degrees equal to a power of q. But the order of a transitive constituent of degree p is p, and would therefore not contain q as a factor, but every prime which divides the order one transitive constituent of G_1 divides the order of each of its transitive constituent.

THEOREM 4.—*If p and q are distinct primes of the form $2^m + 1$, and if G_8 is of odd order, and has as a transitive constituent an imprimitive group of degree pq; then, according as T has p or q systems of imprimitivity, the order of G_s is not divisible by p^2 or q^2.*

To make the conditions definite, suppose that T has q systems of imprimitivity. These systems are then permuted according to the cyclical group of

order q, and all the substitutions in the tail of T are of degree pq. Corresponding to identity in T, there is in G_s an invariant subgroup H_s of degree $n-\alpha\,(\alpha \nless pq+1)$. If we can show that the order of H_s is not divisible by q, our theorem is proved. Let G_r be the conjugate of G_s which leaves fixed an element of T. Also let R_s be the invariant subgroup of G_s corresponding to the head of T. In G_r the subgroup H_s is one of a set of pq conjugates transformed by G_r according to a transitive constituent T_1 of order $p^{\alpha_1}q^{\alpha_2}$. According to the lemma, T_1 is imprimitive and its order is not divisible by both p^2 and q^2. The subgroup T_1, leaving a given letter fixed, would leave more than one letter fixed. Hence in G_r the subgroup H_s is transformed into itself by some of its conjugates. Let H_{s_1} be one of these conjugates such that $H_{s_1}^{-1}H_sH_{s_1}=H_s$. H_{s_1} then occurs in both G_s and G_r. Hence, it occurs in R_s. If R_s contains operators of order q, they clearly occur in H_s. Hence, H_s and H_{s_1} have the same substitutions of order q. But H_s is invariant in G_s and H_{s_1} in G_{s_1}. The substitutions of order q in H_s would then generate a group invariant in both G_s and G_{s_1}. But this is impossible, since G_s is maximal. Hence the theorem.

§3.—*On Certain Subgroups Contained in G.*

Let p^α be the highest power of a prime p which divides the order of G, and suppose that the number p is prime to n, the degree of G. Let P be any subgroup of order p^α. It must be contained in some of the subgroups G_1, G_2, ..., G_n, leaving a given letter fixed, since its degree is prime to n. If P is of degree $n-\lambda\,(\lambda>1)$, it is proved by Burnside ("Theory of Groups," p. 202), that the subgroup of G, which contains all the substitutions of G which transform P into itself, permutes the λ elements not occurring in P transitively. It is our object to consider the case $\lambda=1$. Let P' be a subgroup of order p^β common to any two of the subgroups G_1, G_2, G_3, ..., G_n such that there is no subgroup of order $p^\gamma\,(\gamma>\beta)$ common to any two of these subgroups. We shall first assume $\beta>0$. P' must be contained in subgroups P_1 and P_2 of order p^α in those subgroups which leave a given letter fixed in which it occurs. Since, in a subgroup of order p^α, any subgroup P' is transformed into itself by operators of the group not contained in P', it follows that P' is invariant in a subgroup P'' of P_1 which is of degree $n-1$. Likewise in P_2 the subgroup P' is invariant in a subgroup P''' of degree $n-1$. Hence, the subgroup P' is invariant in $\{P'', P'''\}$

of degree n. Since $n \equiv 1 \bmod p$, the number of elements of G not occurring in P' is congruent to unity mod p. Also, since P'' and P''' are each of degree $n-1$, it follows that $\{P'', P'''\}$ has a transitive constituent of degree $1 + kp\ (k > 0)$, formed of elements not occurring in P', and whose order is multiple of p.

When $\beta = 0$, the subgroup P is clearly formed by establishing a simple isomorphism between regular groups. Hence,

THEOREM 5.—*If p^α is the highest power of a prime p which divides the order of G, and if a subgroup P of order p^α is of degree $n-1$, then, unless P is a regular group or is formed by establishing a simple isomorphism between regular groups of order p^α, G contains an intransitive subgroup of degree n having a transitive constituent of degree $1 + kp\ (k > 0)$ and of order lp.*

COR. *If p^α is the highest power of a prime p which divides the order of G_s, and if the degree of each transitive constituent of G_s is divisible by p^β, but at least one of them is not divisible by $p^{\beta+1}$, then either $\alpha = \beta$ or the group G contains a subgroup of degree n having a transitive constituent of degree $1 + kp\,(k > 0)$ and of order equal to a multiple of p.*

It may be observed that the theorem and corollary just stated apply to any transitive group in which the subgroup which leaves a given letter fixed leaves only one letter fixed, as well as to a primitive group.

§4.—*On the Transitive Constituents of G_s.*

THEOREM 6.—*If G_s has an invariant subgroup H_s of degree $n - \alpha\,(\alpha > 1)$, then G_s has at least one transitive constituent whose degree exceeds the degree of any transitive constituent of H_s.*

Suppose, if possible, that H_s has a transitive constituent T such that its degree is equal to the degree of the transitive constituents of G_s of largest degree.

Consider a conjugate G_r of G_s, leaving fixed an element of G_s not occurring in H_s. Since H_s occurs in both G_s and G_r, these two groups have at least one transitive constituent in the same elements, i. e., in the elements of T. The group $\{G_r, G_s\}$ would then be intransitive. But $\{G_r, G_s\}$ must be identical

with G, since G_s is maximal. Hence the hypothesis that H_s has the transitive constituent T leads to an absurdity.

Cor. *If all the transitive constituents of G_s are primitive groups of the same degree t, then G_s is formed by establishing a simple isomorphism between these transitive constituents.*

This follows readily from the theorem if we remember that every invariant subgroup of a primitive group is transitive.

THEOREM 7.—*If G_s has as a transitive constituent a regular group T of degree t, and if the order of G_s exceeds t, then G_s has another transitive constituent of degree t which has the property that its subgroup which leaves a given letter fixed permutes all the remaining letters.*

Consider the invariant subgroup H_s of G_s corresponding to identity in T. In a conjugate H_r of G_s, leaving fixed an element of T, there occur just $1/t$ of the substitutions of G_s and the subgroup H_s is one of t conjugates transformed according to a transitive constituent T_1. If, in the group T_1, the subgroup which leaves a given letter fixed, leaves more than one letter fixed, H_s is transformed into itself by some of its t conjugates under G_r. But the substitutions of H_s are the only substitutions common to G_r and G_s. Hence, the transitive constituent T_1 has the property mentioned in the theorem.

THEOREM 8.—*If G_s has λ systems of intransitivity, and if H_s is the invariant subgroup of G_s corresponding to identity in any transitive constituent T, while G_s transforms $H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$* (defined as in §2) *according to a constituent group having μ systems of intransitivity, then H_s has more than $\frac{\lambda}{\mu}$ systems of intransitivity, excepting when $\mu = 1$, and then it has at least λ.**

If H_s has as few as $\frac{\lambda}{\mu}$ systems of intransitivity, the μ conjugate sets under G_s into which $H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$ are divided could, at most, contain elements from $\left(\frac{\lambda}{\mu} - 1\right)\mu + 1 = \lambda - \mu + 1$ of the λ systems of intransitivity of G_s, since each of the subgroups $H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$ must contain at least a cycle from T.

* Cf. Miller, loc. cit., p. 535

These subgroups could not then generate a group of degree $n-1$ unless $\mu = 1$. Hence, by means of 1, §2, the theorem follows.

Cor. *If all the transitive constituents of G_s are primitive groups, $H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$ cannot be a single conjugate set under G_s.*

§5.—*On the Transitive Constituents of G_s when the Order of G is Restricted to be an Odd Number.*

Burnside recently proved the interesting theorem* that, if G is of odd order, G_s has its transitive constituents in pairs of the same degree.

Let $a_{s_1}, a_{s_2}, a_{s_3}, \ldots, a_{s_t}$ be the elements of any transitive constituent of degree t. The above theorem was proved by considering the quadratic function

$$f = \sum_{s=1}^{s=n} a_s (a_{s_1} + a_{s_2} + \ldots + a_{s_t}),$$

which is transformed into itself by all the substitutions of G. In this summation, a_s occurs in the parentheses exactly t times. Hence the function f may also be written

$$f = \sum_{s=1}^{s=n} (a_{s'_1} + a_{s'_2} + \ldots + a_{s'_t})\, a_s,$$

and it is shown in the proof of the above theorem that the elements $a_{s'_1}, a_{s'_2}, a_{s'_3}, \ldots, a_{s'_t}$ are elements of a transitive constituent T' of G_s distinct from T. The constituents T and T' will be spoken of as a "pair of transitive constituents." Use will be made of the two ways in which f is written to prove some theorems in reference to the transformation by G_s of its subgroups $H_{s_1}, H_{s_2}, \ldots, H_{s_{\alpha-1}}$ (defined as in §2) when H_s corresponds to identity in T. It is known (p. 5) that these $\alpha - 1$ subgroups are transformed by G_s according to one of its constituent groups. But it is not known whether this constituent group ever contains elements occurring in H_s. Form the conjugate G_{s_α} of G_s, leaving fixed an element of T. From the two ways of writing f, it is seen that in G_{s_α} the element a_s occurs in the transform of T'; i. e., in $R^{-1}T'R$, where R is such that $R^{-1}G_sR = G_{s_\alpha}$. But H_s is transformed by G in the same manner as a_s is replaced. Hence,

* Loc. cit., p. 163.

THEOREM 9.—*Some of the subgroups $H_{s_1}, H_{s_2}, \ldots, H_{s_{a-1}}$ are transformed according to T' when H_s corresponds to identity in T.*

Cor. *If the subgroups $H_{s_1}, H_{s_2}, \ldots, H_{s_{a-1}}$ are a single conjugate set under G_s, they are transformed according to T' when H_s corresponds to identity in T.*

Suppose, next, that G_s has only two transitive constituents T' and T. If, corresponding to identity in one of these constituents, say T, there is in G_s an invariant subgroup H_s, the subgroups $H_{s_1}, H_{s_2}, \ldots, H_{s_{\frac{n-1}{2}}}$ (Cor., Theor. 9) are transformed by G_s according to the elements of H_s. Then, for any two of the n subgroups $H_1, H_2, \ldots, H_n$, which are conjugate under G, one of two relations

$$H_\beta^{-1}H_\alpha H_\beta = H_\alpha \quad \text{or} \quad H_\alpha^{-1}H_\beta H_\alpha = H_\beta \tag{1}$$

holds, but both cannot hold for any two of the subgroups. Let x be the number of elements common to H_α and H_β; then x is clearly the number of elements common to any two of the H's. Also, let $x + y$ be the degree of H_s. Let $a_1, a_2, \ldots, a_y, b_1, b_2, \ldots, b_x$ be the elements of H_s, and $b_1, b_2, \ldots, b_x, c_1, c_2, \ldots, c_y$ the elements of H_{s_1}, one of the subgroups $H_{s_1}, H_{s_2}, \ldots, H_{s_{\frac{n-1}{2}}}$ contained in G_s. Since H_{s_1} must be transformed according to an element of H_s not contained in H_{s_1}, it must be transformed according to one of the a's. There must be substitutions in H_s which do not transform H_{s_1} into itself. If S is such a substitution, then $S^{-1}H_{s_1}S$ contains all the a's, since H_{s_1} and $S^{-1}H_{s_1}S$ have just x elements in common. But since a_{s_1}, according to which H_{s_1} is permuted, occurs in $S^{-1}H_{s_1}S$, this latter subgroup cannot transform H_{s_1} into itself. By exactly the same reasoning H_{s_1} cannot transform $S^{-1}H_{s_1}S$ into itself. But this is contrary to relations (1). Hence,

THEOREM 10.—*If, in a primitive group G of odd order, the subgroup G_s has only two transitive constituents, G_s is formed by establishing a simple isomorphism between them.*

THEOREM 11.—*If, in a primitive group G of odd order, G_s has not more than four transitive constituents, and if these are all primitive groups, then it is formed by establishing a simple isomorphism between them.*

Since G_s has an even number of transitive constituents, we need consider only the cases where it has two or four transitive constituents. Since any invariant subgroup of a primitive group is transitive, and since a simply transitive

primitive group of degree n cannot have a transitive subgroup of degree less than n, the theorem follows at once when G_s has only two transitive constituents.

If G_s has four transitive constituents, and is not formed according to the theorem, there corresponds to identity in some transitive constituent T of degree t an intransitive subgroup H_s invariant in G_s. It has two or three systems of intransitivity. Suppose, first, that H_s of degree $n - \alpha$ has three systems. Then $\alpha - 1 = t$. In the conjugate of G_s, leaving fixed a letter of T, the subgroup H is one of t conjugates. But these $\alpha - 1$ subgroups cannot be conjugate (Cor., Theor. 8). It remains to consider the case where H_s has two systems of intransitivity; then $n - \alpha$ (the degree of H_s) is an even number. Hence $\alpha - 1$ is an even number and the $\alpha - 1$ subgroups H_1, H_{s_2}, ..., $H_{s_{\alpha-1}}$ could only be transformed according to a group T' having two transitive constituents. But by Theor. 8 this is impossible. Hence the theorem.

§6.—*Certain Primitive Groups of Odd Order contained in the Holomorph of the Abelian Group P of Order p^m (p an odd prime) of Type $(1, 1, \ldots, 1)$.*

Represent P as a regular group. Suppose that the order of its group of isomorphisms L is divisible by q^n (q an odd prime). To any subgroup of order q^n in L there corresponds in the holomorph of P a transitive group of degree p^m, and of order $p^m q^n$. The subgroup of this transitive group, which leaves a given letter fixed, is of order q^n, and is clearly maximal, if m is the index to which p belongs mod q. Hence,

THEOREM 12.—*If $p^m \equiv 1 \bmod q^n$ $(n \not< 1)$, m being the index to which p belongs mod q, there is a primitive group G of order $p^m q^n$ contained in the holomorph of the abelian group of order p^m of type $(1, 1, \ldots, 1)$.*

Cor. 1. *If q^n is the highest power of q which divides $p^m - 1$, there is only one group G satisfying the above conditions.*

Cor. 2. *If $p - 1$ $(p \neq 3)$ is not divisible by 3, there exists a primitive group G of degree p^2 and of order $3p^2$. Furthermore, if $p - 1$ is divisible by 3, there is no primitive group of degree p^2 whose order is $3p^2$.*

The first part of this corollary is merely a special case of the general theorem.

The second part may be proved as follows:

By Sylow's theorem, a group of this order contains a single subgroup P of

order p^2. Since G is primitive, an invariant subgroup P must be transitive. The subgroup is, therefore, regular, and it must be the non-cyclical group of order p^2. P would contain $p+1$ subgroups of order p, these would have to occur in conjugate sets of three, since G cannot contain an invariant intransitive subgroup. But $p+1$ is not divisible by 3 when $p-1$ is divisible by 3.

§7.—*On the Class of Primitive Groups G of Odd Order.*

By the class of a substitution group is meant the smallest number of elements in any one of its substitutions besides identity.*

Let $n-\mu$ represent the class of G. For all odd values of μ less than 7 there exist groups G of odd order of class $n-\mu$. Thus:

For $\mu=1$, in any non-cyclic invariant subgroup of a metacyclic group.
For $\mu=3$, in the primitive group of degree 27 of order 27.39.†
For $\mu=5$, in the primitive group of degree 125 of order 125.93.‡

It will now be shown that there is no primitive group of odd order in which μ is even and less than 6. G_s has an even number of transitive constituents in pairs of the same degree (p. 10), and is clearly formed by establishing a simple isomorphism between its transitive constituents. If $\mu=2$, at least two of the transitive constituents must be non-regular, since they are in pairs of the same degree. Suppose that t_1 is the degree of one of these non-regular transitive constituents. It must clearly be of class t_1-1. In the constituent of degree $2t_1$ formed by combining these two, every substitution of degree t_1-1 would correspond to a substitution of degree t_1, or G would contain substitutions of degree $n-3$. But this is clearly impossible, since a transitive group of degree t_1 and of class t_1-1, the order of the substitutions of degree t_1 is prime to the order of those of degree t_1-1.

It remains to consider the case when $\mu=4$. Here again G_s must be formed by establishing a simple isomorphism between transitive constituents, not all of which can be regular. If t_1 is the degree of any non-regular transitive constituent, this constituent must either be of class t_1-3 or t_1-1. Suppose that all

* Jordan, Liouville, Vol. 16 (1871), p. 408.

† Burnside, loc. cit., p. 180.

‡ See p. 30 of this paper.

the non-regular transitive constituents are of class one less than their degrees. Since there must be an even number of such transitive constituents, it readily follows that G_8 cannot have more than two such transitive constituents or G would contain substitutions of degree less than $n-4$. But in this case G would have no substitutions of degree less than $n-3$. Hence G_8 must have at least one transitive constituent T of some degree t_2 of class t_2-3. Now, G_8 must have at least one more transitive constituent T' of degree t_2. This constituent must be of class t_2-1 or t_3-1. In combining T and T' into a constituent of degree $2t_2$, all the substitutions of degree t_2-3 in one must correspond to substitutions of degree t_2 in the other or G would contain substitutions of degree less than $n-4$. From this it is easily seen that substitutions of degree t_2-3 and those of degree t_2 to which they correspond must be regular. Hence all the substitutions of degree t_2-3 would be of order 3. Consider the subgroup P_3 of G_8 corresponding to a subgroup of T of degree t_2-3 of order 3^a such that there is no subgroup of T of order 3^{a+1} which is of degree t_2-3. P_3 would then be invariant in a subgroup of G of degree n, and the 4 letters of G not occurring in P_3 would be transitively connected so that the order of G must be an even number. Hence there is no primitive group of odd order of class $n-4$.

PART II.

§8.—*On the Primitive Groups of Odd Order of Degree less than* 243.

It is known* that all transitive groups of odd order of prime degree are invariant subgroups of the metacyclic group. Inasmuch as these groups are well known, we shall consider only those primitive groups whose degrees are not primes. As already stated in this paper (p. 1), the primitive groups of odd order have been determined for degrees not exceeding 100. It is the object of this part to extend this determination to all degrees less than 243. We are then concerned with the groups whose degrees lie between 100 and 243. It is stated without proof by Burnside (loc. cit., note, p. 185) that any transitive group of odd order of degree $3p$ (p a prime) is imprimitive. We have examined all the com-

* Burnside, loc. cit., p. 177.

posite numbers within the given limits for the degrees of primitive groups of odd order, and the results agree with this statement. We shall, therefore, for the sake of brevity, omit these degrees.

Represent by G^n a primitive group of odd composite order of degree n. As in Part I, let G_s denote the subgroup of G^n containing all the substitutions which leave a given letter a_s fixed. The method is, briefly, as follows:

For each odd degree n (n not a prime nor 3 times a prime) it is assumed that a group G^n exists. The degree of any solvable primitive group is a power of a prime.* Hence, in order to prove that no group G^n exists, for a given value of n which is not the power of a prime, it is sufficient to prove that there is no simple group G^n, provided there is no simple group of odd composite order of degree less than n. As the latter condition is satisfied, it is further assumed that G^n is simple when it is not a power of a prime.

We write down for examination all those, and only those, systems of intransitivity of G_s which are not excluded by the conditions, 1°, that every prime which divides the order of one transitive constituent divides the order of every transitive constituent;† 2°, that the transitive constituents occur in pairs of the same degree;‡ 3°, that when G^n is simple, there can be no transitive constituent whose degree is a prime of the form $2^m + 1$;§ 4°, that if the degree of one transitive constituent is a prime of the form $2^m + 1$, all the transitive constituents are of this degree.

This method of excluding transitive constituents of G_s depends, of course, on a knowledge of the primes which occur in the orders of transitive groups of odd order of degree less than $n/2$. The following table shows the primes which may occur in the orders of transitive groups of odd order of degree less than 120. The primes written under the degree are the primes which occur in the orders of some transitive groups of odd order of the given degree in addition to those primes contained in the degree itself.

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* Lettre de Galois, a M. Auguste Chevalier, Liouville's Journ. (1846), p. 41.

† Jordan, loc. cit., p. 284. ‡ Burnside, loc. cit., p. 165. § Miller, loc. cit., p. 6.

Degree:	3	5	7	9	11	13	15	17	19	21	23
Primes:			3		5	3			3		11
Degree:	25	27	29	31	33	35	37	39	41	43	45
Primes:	3	13	7	3, 5	5	3	3		5	3, 7	
Degree:	47	49	51	53	55	57	59	61	63	65	67
Primes:	23	3		13			29	3, 5		3	3, 11
Degree:	69	71	73	75	77	79	81	83	85	87	89
Primes:	11	5, 7	3		3, 5	3, 13	5, 13	41		7	11
Degree:	91	93	95	97	99	101	103	105	107	109	111
Primes:	3	5	3	3	5	5	3, 17		53	3	
Degree:	113	115	117	119							
Primes:	7	11		3.							

We shall use $S_1, S_2, \ldots, S_t$ to represent the sets of systems of intransitivity of G_s which are not excluded by the conditions above stated. The transitive constituents thus obtained are either shown to lead to impossibilities or the orders and number of the groups are determined.

§9.—*Determination of the Groups* G^n $(100 < n < 243)$.

G^{105}.

$$S_1 = 13, 13, 13, \ldots, 13. \qquad S_2 = 13, 13, 39, 39.$$

If G_s has systems S_1, the order of G^{105} would, by Cor. 2, Theor. 3, be $5 \cdot 13 \cdot 3^{\alpha} \cdot 7$ $(\alpha \not> 2)$. But G^{105} would then contain in its order less than 7 prime factors, and could not be a simple group.* If G_s has systems S_2, the order of G would be $5 \cdot 13 \cdot 7 \cdot 3^{\beta}$ (Cor. 1, Theor. 3). Let $p = 5$ in Cor. 2, Theor. 1, and it follows that G^{105} does not exist.

G^{115}.

$$S_1 = 19, 19, \ldots, 19. \qquad S_2 = 57, 57.$$

G^{115} would be of order $3^{\alpha} \cdot 5 \cdot 19^{\beta} \cdot 23$. Let $p = 23$ in Cor. 2, Theor. 1, and it follows that G^{115} does not exist.

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* Burnside, loc. cit., pp. 265–268.

G^{117}.

$$S_1 = 29,\ 29,\ 29,\ 29.$$

The order of G_s would be, by Cor. 2, Theor. 3, $29 \,.\, 7^{\alpha}$ ($\alpha \not> 1$). But G^{117} would then contain in its order less than 7 primes, and could therefore not be a simple group.

G^{119}.

$$S_1 = 59,\ 59.$$

G^{119} would be of order $17 \,.\, 7 \,.\, 59^{\alpha} \,.\, 29^{\beta}$. Let $p = 17$ in Cor. 2, §1, and it follows that G^{119} does not exist.

G^{121}.

$$S_1 = 3,\ 3,\ \ldots\ldots,\ 3. \qquad S_3 = 15,\ 15,\ \ldots\ldots,\ 15.$$
$$S_2 = 5,\ 5,\ \ldots\ldots,\ 5. \qquad S_4 = 15,\ 15,\ 45,\ 45.$$

If G_s has systems S_1 the order of G^{121} would be $121 \,.\, 3$.* By Cor. 2, §6, there is a group of order $3 \,.\, 121$, and it is easily seen that there is only one such group. With S_2, the order of G^{121} would be $121 \,.\, 5$. Any group of this order contains at least one invariant subgroup of order 11. As this subgroup would be intransitive, it cannot occur in a primitive group.

If G_s has systems S_3 or S_4, the order of G^{121} would be of the form $11^2 \,.\, 3^{\alpha} \,.\, 5^{\beta}$ If any transitive constituent of degree 15 has 5 systems of imprimitivity the order of G_s is not divisible by 5^2 (Theor. 4). G^{121} would then be of order $11^2 \,.\, 3^{\alpha} \,.\, 5$. The subgroup of order 3^{α} would then be of degree 120. If $\alpha > 1$, from §3, G^{121} would contain an intransitive subgroup having a transitive constituent of degree $1 + 3k$ ($k > 1$), and whose order is divisible by 3. The number of the form $1 + 3k$ could only be 55 and be a divisor of the order. But no transitive group of degree 55 of odd order has its order divisible by 3. The transitive constituents of degree 15 must then all have 3 systems of imprimitivity, and G^{121} would be of order $121 \,.\, 3 \,.\, 5^{\beta}$. If $\beta > 1$, it follows from §3 that a subgroup P of greatest order common to two subgroups of G^{121}, leaving given letters fixed, is invariant in a subgroup of order $11^2 \,.\, 5^{\beta}$, $11 \,.\, 3 \,.\, 5^{\beta}$ or $11 \,.\, 5^{\beta}$. The conjugate set to which

* Miller, loc. cit., p. 536.

P belongs under G^{121} would then be transformed by G^{121}, according to a transitive group of degree 3, 11 or 33. Now, it is easily seen that $\beta \not> 2$. We shall show that one group G^{121} exists when $\beta = 1$, and none when $\beta = 2$.

A group of order $11^2 . 5^\beta . 3$ $(\beta \not> 2)$ must, by Sylow's theorem, contain an invariant subgroup P_{121} of order 11^2. This subgroup in G^{121} must be of type (1, 1), since the cyclical group would contain a single subgroup of order 11 which would be an invariant intransitive subgroup of G^{121}. The group G^{121} must then occur in the holomorph of P_{121}. The group of isomorphisms L of P_{121} is of order $120 . 110 = 3 . 5^2 . 2^4 . 11$. Now, L contains just 55 subgroups* of order 3, each being invariant in a subgroup of order $3 . 5 . 2^4$. Hence L contains subgroups of order 15 and they are all conjugate. It results, therefore, that the transitive groups of order $121 . 15$ in the holomorph of G_{121} are conjugate. The subgroup of order 15 which leaves a giving letter fixed, is clearly maximal. Hence the group is primitive. Similarly, examining L, it is found to contain no subgroups of order $3 . 5^2$. Hence there is no group G^{121} of order $11^2 . 3 . 5^2$ contained in the holomorph of P^{121}.

G^{125}.

$$S_1 \doteq 31, 31, 31, 31.$$

The order of G^{125} would be $5^\alpha . 31 . 3^\beta$ ($\alpha = 3$ or 4, $\beta = 0$ or 1, Cor. 2, §2). By Sylow's theorem, G^{125} would contain 1 or 31 subgroups of order 5^α. If it contained only one, $\alpha = 3$; for, if $\alpha = 4$, G_8 and its conjugates would contain more than 5^4 substitutions of order 5. If it contained 31 subgroups of order 5^α, they would be transformed by G^{125} according to a non-cyclic transitive group of degree 31 isomorphic with G^{125}. The subgroup of G^{125} corresponding to identity on this quotient group would contain a single subgroup of order 5^3. Any group G^{125} then contains an invariant subgroup P_{125} of order 125. The group P_{125} must be the abelian group of type (1, 1, 1), since all other groups of order 125 contain characteristic subgroups, and a characteristic subgroup of P_{125} would be an invariant intransitive subgroup of G^{125}. The group G^{125} is then contained in the holomorph of P_{125}. The group of isomorphisms L of P_{125} is of order $2^7 . 3 . 5^3 . 31$. The group L contains, by Sylow's theorem, 1, 2^5, 5^3 or $2^5 . 5^3$ subgroups of order

* This is shown from the composition series of L. The factor groups of L are the cyclic group of order 10, the simple group of order 660 representable on 11 letters, and the group of order 2.

31. That it could not contain 1 or 2^5 such subgroups is easily seen from the fact that L occurs as a transitive group of degree 124. In L a subgroup of order 31 is then invariant in a subgroup of order $31 . 2^2 . 3$ or $31 . 2^7 . 3$. Hence L contains subgroups of order 31 and 31 . 3, but it does not contain any subgroup of order 31 . 5 or 31 . 15. Corresponding to these subgroups, there are in the holomorph of P_{125} transitive groups of orders $5^3 . 31$ and $5^3 . 31 . 3$. These groups are evidently primitive groups. That there is only one group G^{125} of each of these orders follows from the fact that in L the subgroups of each of the orders 31 and 31 . 3 form a single conjugate set.

G^{133}.

$S_1 = 11, 11, \ldots, 11.$ $\quad$ $S_4 = 27, 27, 39, 39.$

$S_2 = 11, 11, 55, 55.$ $\quad$ $S_5 = 27, 27, 13, 13, \ldots, 13.$

$S_3 = 33, 33, 33, 33.$

If G_s has systems S_1 or S_2, the order of G^{133} would be of the form $19 . 7 . 11^\alpha . 5^\beta$. Let $p = 19$ in Cor. 2, §1, and it follows at once that G^{133} does no exist in the case under consideration.

If G_s has systems S_3, the order of G^{133} would be $19 . 7 . 11^\alpha . 5^\beta . 3^\gamma$. From §3 it follows that $\alpha = 1$. Now, it is easily seen that an imprimitive group of odd order of degree 33 whose order is not divisible by 11^2, does not have its order divisible by 5^2. Hence, the order of any transitive constituent T would be $11 . 3^{\alpha_1} . 5^{\beta_1}$. ($\beta_1 \not> 1$). Corresponding to identity in T, there could be no substitutions of order 5 or there would also be substitution of order 11, and the order of G_s would be divisible by 11^2. Hence, $\beta \not> 1$. If $\beta = 1$, G^{133} contains an invariant subgroup of index 5.* If $\beta = 0$, G^{133} contains an invariant subgroup of index 11. Hence, G^{133} does not exist if G_s has systems S_3.

If G_s has systems S_4 or S_5, the order of G^{133} is of the form $7 . 19 . 3^\alpha . 13^\beta$. The transitive constituents of degree 27 must be primitive groups. Let T_1 and T_2 represent the transitive constituents of degree 27. Consider the invariant subgroup H_s of G_s corresponding to identity in T_1. If any elements of T_2 occur in H_s, the latter must have a transitive constituent of degree 27, which is an invariant subgroup of T_2. Now, it is easily seen that an imprimitive group of degree 39 of odd order cannot contain a transitive subgroup of degree 27. Hence a conjugate G_r of G_s in which H_s occurs would contain a transitive constituent of

* Burnside, loc. cit., pp. 261–262.

degree 27 having the same elements as T_2. $\{G_r, G_s\}$ would then be an intransitive group. But $\{G_r, G_s\}$ must coincide with G^{133}, since G_s is maximal. Hence H_s has no transitive constituent of degree 27.

T_1 and T_2 must then be combined by establishing a simple isomorphism between them, and H_s could only be of degree $\overline{\gtrless} 39 . 2$. From the fact that $2 . 27$ of the conjugates of H_s under G^{133} contained in G_s would be transformed according to the constituent of degree 54 in G_s, this assumption as to the degree of H^s readily leads to impossibilities. Hence, the order of G_s is equal to the order of its transitive constituent of degree 27. The order of G^{133} would then be $133 . 3^\alpha . 13$ ($\alpha \not> 4$), but a group of this order could not, by Sylow's theorem, contain more than 39 subgroups of order 19. Hence G^{133} does not exist.

G^{135}.

$S_1 = 67, 67.$ $\qquad$ $S_2 = 27, 27, 27, 27, 13, 13.$

If G_s has systems S_1, the order of G^{135} would be $67 . 5 . 3^\alpha . 11^\beta$ ($\alpha \not> 4$, $\beta = 0$ or 1, Cor. 2, Theor. 3). Let $p = 5$ in Cor. 2, Theor. 1, and it follows that G^{135} does not exist. If G_s has systems S_2, the order of G^{135} is $5 . 3^\alpha . 13$. By Sylow's theorem, a group of this order contains not more than 13 subgroups of order 3^α, Hence G^{135} does not exist.

G^{143}.

$S_1 = 71, 71.$

G^{143} would be of order $11 . 13 . 71 . 5^\alpha . 7^\beta$ ($\alpha \overline{\gtrless} 1$, $\beta \overline{\gtrless} 1$) Cor. 2, Theor. 3. As this order would be the product of distinct primes, G^{143} cannot be a simple group* and therefore does not exist.

G^{145}.

$S_1 = 9, 9, \ldots, 9.$ $\qquad$ $S_2 = 9, 9, \ldots, 9, 27, 27.$

$S_3 = 9, 9, \ldots, 9, 27, 27, 27, 27.$

The group G^{145} would be of order $29 . 5 . 3^\alpha$. Let $p = 29$ in Cor. 2, Theor. 1 and it follows that G^{145} does not exist.

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* Frobenius, Berliner Sitzungsberichte (1893), p. 337.

G^{147}.

$$S_1 = 73,\ 73.$$

By Cor. 2, Theor. 3, the order of G^{147} would be $7^2 . 73 . 3^{\alpha}$ ($\alpha \not> 3$). But a group of this order would, by Sylow's theorem, contain a single subgroup of order 7^2. As this subgroup would be an invariant intransitive subgroup, no group G^{147} exists.

G^{153}.

$$S_1 = 19,\ 19,\ \ldots\ ,\ 19. \qquad S_2 = 19,\ 19,\ 57,\ 57.$$

The group G^{153} would be of order $17 . 3^{\alpha} . 19^{\beta}$. If in Cor. 2, §1, we make $p = 17$, it follows that G^{153} does not exist.

G^{155}.

$$S_1 = 11,\ 11,\ \ldots\ ,\ 11. \qquad S_2 = 11,\ 11,\ 11,\ 11,\ 55,\ 55. \qquad S_3 = 77,\ 77.$$

$S_4 =$ any set of systems in which one system contains 7 letters. If G_s has systems S_1 or S_2, its order is not divisible by 11^2 (§2, Cor.) The order of G^{155} would then be $11 . 31 . 5^{\alpha}$. By Sylow's theorem, a group of this order would contain 1, 11, 31 or 341 subgroups of order 5. It could not contain 1, 11, 31 and be a simple group, since there is no simple group of odd composite order of degree < 155. If G^{155} contains 341 subgroups of order 5^{α}, it occurs as a primitive group of degree 341. Since 340 is not divisible by 5^2, the order of the subgroup which leaves a given letter fixed would not exceed 5. But a group of order $31 . 11 . 5$ is clearly solvable. If G_s has systems S_3, the order of G^{155} is of the form $31 . 5^{\alpha} . 7^{\beta} . 11^{\gamma} . 3^{\delta}$. Let $p = 11$ and 7 in Cor., §3, and it follows that $\gamma = 1$ and $\beta = 1$. Since G_s has only two transitive constituents, it follows that the order of G_s is equal to that of one of its transitive constituents. But if the order of a transitive group of odd order of degree 77 is not divisible by 11^2 nor 7^2, its order is not divisible by 5^2 nor 3^2. Hence $\alpha \leqq 2$ and $\delta \leqq 1$.

If G_s has systems S_4, the order of G^{155} is of the form $5 . 31 . 7^{\alpha} . 3^{\beta}$ and G^{155} contains an invariant subgroup of index 5. If G^{155} exists, its order would then be of the form $3^{\alpha_1} . 5^{\alpha_2} . 7 . 11 . 31$ ($\alpha_1 = 0$ or 1, $\alpha_2 \not> 2$), but a group of this order cannot be a simple group as this number contains less than 7 prime factors. Hence G^{155} does not exist.

G^{161}.

$S_1 = 15, 15, 15, 15, 25, 25, \ldots\ldots, 25$. $\quad S_2 = 27, 27, 27, 27, 13, \ldots\ldots, 13$.

The order of G^{161} would be of the form $7.23.5^{\alpha}.5^{\beta}.13^{\gamma}$. Let $p = 23$ in Cor. 2, Theor. 1, and it follows that G^{161} does not exist.

G^{165}.

$S_1 = 41, 41, 41, 41$.

By Cor. 2, Theor. 3, the order of G^{165} would be $11.5^{\alpha}.3.41$ ($\alpha \ngtr 2$), and G^{165} would, by Sylow's theorem, contain a single subgroup of order 11. Hence G^{165} does not exist.

G^{169}.

$S_1 = 3, 3, \ldots\ldots, 3$.
$S_2 = 7, 7, \ldots\ldots, 7$.
$S_3 = 7, 7, \ldots\ldots, 7, 21, 21$.
$S_4 = 7, 7, \ldots\ldots, 7, 21, 21, 21, 21$.
$S_5 = 7, 7, \ldots\ldots, 7, 21, 21, 21, 21, 21, 21$.
$S_6 = 21, 21, \ldots\ldots, 21$.
$S_7 = 21, 21, 63, 63$.
$S_8 = 7, 7, \ldots\ldots, 7, 49, 49$.
$S_9 = 7, 7, 7, 7, 21, 21, 49, 49$.
$S_{10} = 7, 7, \ldots\ldots, 7, 63, 63$.

If G_s has systems S_1, the order of G^{169} is $13^2.3$, but by Cor. 2, §6, there is no group G^{169} of this order.

If G_s has some transitive constituent of degree 7, the order of G^{169} is of the form $13^2.7.3^{\alpha}$ (Cor. 2, Theor. 3). G^{169} would contain 13, 91 or 169 subgroups of order 3^{α}, if $\alpha > 0$. For, since the number of such subgroup in G_s must be 7, the total number in G^{169} is $\frac{7.169}{\lambda}$, where λ is the number of letters of G^{169} left fixed by a subgroup of order 3^{α}, and $\lambda > 1$.

If the number is 13 or 91, by considering the isomorphic group of degree 13 or 91, according to which the conjugate set is transformed, it is easily seen that $\alpha \ngtr 3$. If the number is 169, a subgroup of order 3^{α} is invariant in a subgroup K of order $3^{\alpha}.7$ and of degree 169, since in any subgroup leaving a given letter fixed, a subgroup of order 3^{α} is one of 7 conjugates. Since a substitution of order 7 in G^{169} is of degree 168, every transitive constituent of K has its order divisible by 7. Now there is no transitive group of odd order of degree 3^{β} when $3^{\beta} < 169$, which contains in its order the factor 7. Hence the degree of every transitive constituent of K is a multiple of 7. But 7 is not a divisor of 169.

Hence there cannot be 169 subgroups of order 3^α. We have then shown that α is not greater than 3 when G_s has a transitive constituent of degree 7.

In all the remaining cases G^{169} would be of order $13^2.7^\alpha.3^\beta$. Let $p=7$ in Cor., §3, and it follows that $\alpha=1$. If T is a subgroup such that there is no subgroup of greater order common to two subgroups of order 3^β, then T is of order $3^{\beta-1}$. From the reasoning of §3, it follows that T is invariant in a subgroup of degree 169 of one of the following orders: $13^2.3^\beta$, $13.7.3^\beta$, 13.3^β or 7.3^β. The subgroup T would then be one of 7, 13, 91, or 169 conjugates in G^{169} and just as before it follows that $\beta \ngtr 3$.

It remains to consider the group G^{169} of order $13^2.7.3^\beta$ $(\beta \ngtr 3)$. By Sylow's theorem a group of this order contains 1 or 27 subgroups of order 13^2. It could not contain 27; for, on account of the limitations on β, they would be transformed according to a regular group of degree 27. Hence G^{169} contains a single subgroup order 13^2, and is contained in the holomorph of the abelian group P of order 13^2 of type (1, 1). The group of isomorphisms L of P is of order $2^5.3^2.7.13$. It remains to examine L for subgroups of order 7.3^β $(\beta \ngtr 3)$. Such a subgroup contains a single subgroup of order 7. The group L contains just 78 subgroups of order 7.* Each of these is then invariant in a subgroup of order $7.3.2^4$. From this† it follows that L contains subgroups of order 7 and 21 but none of order 7.3^β $(\beta > 1)$. The subgroups of order 21 are conjugate in L. Hence there is in the holomorph of P just one subgroup of each of the orders 169.7 and 169.21. The subgroup leaving a given letter fixed is in each case maximal. Hence the groups are primitive.

G^{171}.

$S_1 = 15, 15, \ldots, 15, 25, 25.$ $\quad S_2 = 15, 15, 45, 45, 25, 25.$ $\quad S_3 = 85, 85.$

If G_s has systems S_1 or S_2, by Theor. 4, the order of G_s cannot be divisible by 3^3, since the order must be divisible by 5^2 on account of the transitive constituent of degree 5^2. The order of G^{171} is then of the form $19.3^3.5^\alpha$. Let $p=5$ in Cor., §3, and it follows that $\alpha=1$. As a group of $19.3^3.5$ contains a single subgroup of order 19, there is no simple group G in which G_s has systems S_1 or S_2.

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* Shown by considering the composition series of L. The factor groups of L are the cyclical group of order 12, the simple group of order 1092, and the group of order 2. † Ibid.

If G_s has systems S_3, by Theor. 4, the order of G^{171} would be $19.3^2.5.17^{\alpha}$ or $19.3^2.17.5^{\beta}$. But groups of these orders cannot be simple.* Hence G^{171} does not exist.

G^{175}.

$S_1 = 29, 29, \ldots\ldots, 29. \qquad S_2 = 87, 87.$

If G_s has systems S_1, by Cor. 2, Theor. 3, the order of G^{175} would be $7^{\alpha}.5^2.29\ (\alpha \not> 2)$. But a group of this order cannot be a simple group, since it contains not more than 5 prime factors. If G_s has systems S_2, the order of G^{175} is $7^{\alpha}.5^2.29^{\beta}.3^{\gamma}$. Let $p = 29$ in Cor., §3, and it follows that $\beta = 1$. Since G_s has only two transitive constituents, its order is equal to the order of each of its constituents (Theor. 10). Making use of this fact, it is easily seen that 7 cannot occur to a higher power in the order of G_s than the power to which 29 occurs. Hence $\alpha \not> \beta + 1$. If either transitive constituent of G_s has 3 systems of imprimitivity, the order of G_s^{175} is not divisible by 3^2. But the order of G^{175} would then contain not more than 6 prime factors. Hence each transitive constituent must have 29 systems of imprimitivity. G_s would then contain an invariant subgroup of order 3^{γ}. Now the subgroup of such an imprimitive group which leaves a given letter fixed would leave 3 letters fixed. The subgroup K, containing all the substitutions common to G_s and any one of its conjugates G_r, is of order $7^{\alpha-1}\,3^{\gamma-1}$ and is invariant in subgroups of G_s and G_r whose orders are equal to $7^{\alpha-1}.3^{\gamma}$ and which contain a single subgroup of order 3^{γ}. The subgroup of G^{175} which contains all the substitutions which transform K into itself contains $1 + 3k\,(k > 0)$ subgroups of order 3^{γ}. Its order could then only be $7^{\alpha}.3^{\gamma}$ or $7^{\alpha-1}\,5^2.3^{\gamma}$ (other assumptions lead to transitive representations of G^{175} of degree less than 175). If subgroups of these orders occur in G^{175}, it can be represented as a transitive group of degree 725 or 203. Since any divisor of these numbers is less than 175, these representations would be primitive. As a group of degree 725 the subgroup which leaves a given letter fixed would be of order $7^{\alpha}.3^{\gamma}$. As it is easily shown that there is no transitive group of odd order of degree $3^{\delta}\,(\delta < 6)$ whose order contains the factor 7, the degree of each transitive constituent of above subgroup would be a multiple of 7. But 7 is not a divisor of 724. As a group of degree 203, the subgroup which leaves a given letter fixed would be of

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* Burnside, loc. cit., p. 262.

order $7^{\alpha-1}.5^2.3^\gamma$. But no intransitive group of degree 202 of this order can be constructed in which every prime which divides the order of one transitive constituent divides the order of every transitive constituent, and in which the transitive constituents are in pairs of the same degree.

G^{185}.

$S_1 = 23, 23, \ldots, 23.$ $S_2 = 27, 27, 13, 13, \ldots, 13.$

$S_3 = 27, 27, 39, 39, 13, 13, \ldots, 13.$

The order of G^{185} would be of the form $5.37.23^\alpha.11^\beta.3^\gamma.13^\delta$. But a group of this order cannot be simple.*

G^{187}.

$S_1 = 31, 31, 31, \ldots, 31.$ $S_3 = 93, 93.$

$S_2 = 27, 27, 27, 27, 13, 13, \ldots, 13.$ $S_4 = 27, 27, \ldots, 27, 39, 39.$

The order of G^{187} is of the form $17.11.3^\alpha.31^\beta.13^\gamma$. Let $p = 17$ in Cor. 2, §1, and it follows that G^{187} does not exist.

G^{189}.

$S_1 = 47, 47, 47, 47.$ $S_2 = 13, 13, 27, 27, \ldots, 27.$

$S_3 = 13, 13, 81, 81.$

If G_s has systems S_1, the order of G^{189} is of the form $7.3^3.47.23^\alpha$ ($\alpha = 0$ or 1). As the order contains at most 6 prime factors, the group cannot be simple. If G_s has systems S_2 or S_3, the order of G^{189} is $7.13.3^\alpha$ (Cor. 2, §2). By Sylow's theorem, a group of this order contains not more than 91 subgroups of order 3. The group G^{189} could then occur on as few as 91 letters, but this has been shown impossible. Hence G^{189} does not exist.

G^{195}.

$S_1 = 97, 97.$

The group G^{195} would be of order $5.13.3^\alpha.97$ (Cor. 2, Theor. 3). Let $p = 5$ in Cor. 2, §1, and it follows that G^{195} does not exist.

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* Burnside, loc. cit., p. 170.

G^{203}.

$$S_1 = 101, 101.$$

By Cor. 2, Theor. 3, the order of G^{203} would be $7.29.101.5^{\alpha}$ ($\alpha \ngtr 2$). As this number contains not more than 5 prime factors, the group cannot be simple. Hence G^{203} does not exist.

G^{205}.

$$S_1 = 51, 51, 51, 51.$$

The order of G^{205} would be of the form $5.41.3^{\alpha}.17^{\beta}$. Let $p = 5$ in Cor. 2, §1, and it follows that G^{205} does not exist.

G^{207}.

$$S_1 = 103, 103.$$

The order of G^{207} would be of the form $23.3^{\alpha}.103.17$ (Cor. 2, Theor. 3). Let $p = 23$ in Cor. 2, §1, and it follows that G^{207} does not exist.

G^{209}.

$$S_1 = 13, 13, \ldots, 13. \qquad S_2 = 13, 13, \ldots, 13, 39, 39.$$
$$S_3 = 13, 13, 13, 13, 39, 39, 39, 39.$$

If G_s has systems S_1 or S_2, it is easily seen that the order of G^{209} is equal to or a divisor of $11.19.13.3$; but a group of this order cannot be simple, as the order is the product of distinct primes. If G_s has systems S_3, the order of G^{209} is $11.19.13^{\alpha}.3^{\beta}$. Let $p = 11$ in Cor. 2, §1, and it follows that G^{209} does not exist.

G^{215}.

$$S_1 = 107, 107. \qquad S_2 = 13, 13, 13, 13, 27, 27, \ldots 27.$$
$$S_3 = 13, 13, \ldots, 13, 81, 81.$$

The order of G^{215} would be of the form $5.43.107^{\alpha}.53^{\beta}.13^{\gamma}.3^{\delta}$. But a group of this order cannot be simple.* Hence G^{215} does not exist.

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* Burnside, loc. cit., p. 170.

G^{217}.

$S_1 = 9, 9, \ldots, 9.$

$S_2 = 9, 9, \ldots, 9, 27, 27.$

$S_3 = 9, 9, \ldots, 9, 81, 81.$

$S_4 = 9, 9, \ldots, 9, 27, 27, 27, 27.$

$S_5 = 27, 27, 27, 27, 27, 27, 9, 9, \ldots, 9$

$S_6 = 27, 27, \ldots, 27.$

$S_7 = 27, 27, 81, 81.$

For all these systems, except when in S_6 and S_7, the transitive constituents of degree 27 are primitive groups, the order of G_8 is a power of 3. The order of G^{217} would then be $31.7.3^a$, with the exception just mentioned. From the argument of §3, it readily follows that, if $a > 3$, G^{217} would contain a subgroup of order $\overline{\gtrless}\, 3^{a-2}.7$, which order contains a factor $\equiv 1 \bmod 3$. If the order of the subgroup were greater than this number, such a subgroup would lead to a transitive representation of the simple group on less than 217 letters. Hence the subgroup, if G^{217} exists, is of order $3^{a-2}.7$. The simple group would then occur as a transitive group of degree $\dfrac{31.7.3^a}{7.3^{a-2}} = 279$. When represented on 279 letters the group would be primitive and the subgroup which leaves a given letter fixed would be of order 7.3^{a-2}. The degrees of all transitive constituents would be multiples of 7. But 7 is not a divisor of 278.

It remains to consider the case where G_8 has for some of its transitive constituents primitive groups of degree 27. In this case all the transitive constituents are either primitive groups of degree 27 or two of them are imprimitive groups of degree 81. In the former case, by Cor., Theor. 6, §4, the order of G^{217} would be $7.31.3^a.13$ $(a \not> 4)$. But, by Sylow's theorem, a group of this order contains no more than 63 subgroups of order 31. But this is impossible, since the simple group of odd order would occur of degree 63. In the latter case, we shall also show that the order G_8 cannot exceed that of one of the transitive constituents of degree 27. Let T' be the transitive constituent of degree 27 distinct from T. Let H_8 be the invariant subgroup of G_8 corresponding to identity in T, where T is so selected that its order is not greater than that of T'. Then, by Theor. 9, 27 of the subgroups $H_{8_1}, H_{8_2}, \ldots, H_{8_{a-1}}$ (defined in §2) are transformed by G_8 according to T'. These 27 subgroups can contain no elements contained in T', or H_8 would have some transitive constituents of degree 13, which is impossible. Hence, these 27 subgroups generate a subgroup contained in the invariant subgroup of G_8 corresponding to identity in T'. But this is impos-

sible, since, by hypothesis, the order of T' is equal to or greater than the order of T.

G^{221}.

$S_1 = 11, 11, \ldots, 11.$
$S_2 = 11, 11, \ldots, 11, 55, 55.$
$S_3 = 55, 55, 55, 55.$
$S_4 = 15, 15, \ldots, 15, 25, 25, 25, 25.$
$S_5 = 15, 15, 25, 25, 25, 25, 45, 45.$

If G_s has systems S_1, the order of G^{221} is, by Cor. 2, Theor. 3, $17.13.11.5^\alpha$ $(\alpha \not> 1)$. But a group of this order cannot be simple, since it is the product of distinct primes.* If G_s has any of the other systems, the order of G^{221} is $17.13.5^\alpha.3^\beta.11^\gamma$. Let $p = 17$ in Cor. 2, §1, and it follows that G^{221} does not exist.

G^{225}.

$S_1 =$ any set of systems in which some systems contain 7 letters.
$S_2 = 21, 21, \ldots, 21, 49, 49.$ $S_3 = 63, 63, 49, 49.$

If G_s has some transitive constituents of degree 7, the order of G^{225} is of the form $5^2.3^\alpha.7$ (Cor. 2, Theor. 3). But, by Sylow's theorem, a group of this order cannot contain more than 175 subgroups of order 3^α. If G_s has systems S_2 or S_3, the order of G^{225} is of the form $5^2.3^\alpha.7^\beta$. Let $p = 7$ in Cor., §3, and it follows that $\beta = 1$. Then as above G^{225} would contain no more than 175 subgroups of order 3^α. Since there is no simple group of odd composite order of degree 175 the group G^{225} does not exist.

G^{231}.

$S_1 = 15, 15, 25, 25, \ldots, 25.$
$S_2 = 15, 15, 25, 25, 75, 75.$
$S_3 = 15, 15, \ldots, 15, 25, 25.$
$S_4 = 15, 15, \ldots, 15, 25, 25, 45, 45.$
$S_5 = 45, 45, 45, 45, 25, 25.$
$S_6 = 23, 23, \ldots, 23.$
$S_7 = 115, 115.$

If G_s has some transitive constituent of degree 15, the order of G_s is not divisible by 3^2 (Theor. 4). Then G^{231} would be of order $3^2.7.11.5^\alpha$ and would contain an invariant subgroup of index 3^2.† If G_s has systems S_6 or S_7, the

* Frobenius, loc. cit., p. 337.

† Burnside, loc. cit., pp. 260, 262.

order of G^{231} would be of the form $3.7.11^{\alpha}.23^{\beta}.5^{\gamma}$; but a group of this order has an invariant subgroup of index 3.

If G_s has systems S_5, the constituents of degree 25 are the primitive group of order 75. As this group is of class 24, we have, by Cor. 3, Theor. 3, that the order of G_s is not divisible by 5^3. The group G^{231} would then be of order $3^{\alpha}.5^2.7.11$. By Sylow's theorem, a group of this order which does not contain less than 231 subgroups of order 3^{α}, contains 385 such subgroups. A simple group of this order would then occur as a primitive group of degree 385. The subgroup G_1, leaving a given letter fixed, would then be of order 5.3^{α}. The only systems of intransitivity of this subgroup G_1, which are not excluded by the conditions stated on p. 15, are 81, 81, 81, 81, 15, 15, 15, 15. In order that the transitive constituents of degree 81 contain in their orders the factor 5, they must be primitive groups of degree 81 of order 81.5. Consider the invariant subgroup H of G_1 corresponding to identity in a transitive constituent of degree 81. It follows from Theor. 6 that the degree of H cannot exceed 60. As G_1 could contain no other similar subgroup, the order of G_1 would be 81.5. The primitive group of degree 385 of order $3^4.5^2.7.11$ then contains 324.77 substitutions of order 5 of degree less than 385. By Sylow's theorem, a group of order $3^4.5^2.7.11$ cannot contain more than 11.81 subgroups of order 5^2. As this number of subgroups contains less than 324.77 substitutions, we have arrived at an absurdity.

G^{235}.

$S_1 = 117, 117.$

$S_2 = 39, 39, \ldots, 39.$

$S_3 = 13, 13, \ldots, 13, 39, 39, 39, 39.$

$S_4 = 13, 13, \ldots, 13, 39, 39.$

$S_5 = 13, 13, \ldots, 13.$

$S_6 =$ any set of systems of which some contain 9 letters.

The order of G^{235} would be of the form $5.47.3^{\alpha}.13^{\beta}$. Let $p = 47$ in Cor. 2, §1, and it follows that G^{235} does not exist.

This completes the determination of the primitive groups of odd order of degree less than 243. The results may be summarized as follows:

Aside from the invariant subgroups of the metacyclic groups there are only ten primitive groups of odd order of degree less than 243. The following list of

numbers gives the orders of these groups, the first factor as they are written being the degree of the group of the given order:* 25.3, 27.13, 27.39, 81.5, 121.3, 121.15, 125.31, 125.93, 169.7, 169.21.

Each of these groups is solvable. The following theorem may then be stated:

THEOREM 13.—*A simple group of odd composite order cannot be of degree less than* 243.

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* For the first four groups, see Burnside, Proc. Lond. Math. Soc., Vol. 33, pp. 178-185.

Theorems on Cardinal Numbers.

By A. N. Whitehead.

In this paper it is proved ($\star 2$) that for cardinal numbers (finite or infinite) if $\alpha < \alpha'$ and $\beta < \beta'$, then $\alpha + \beta < \alpha' + \beta'$. It is already known (cf. $\star 4 \cdot 37$ in my article on *Cardinal Numbers*, in this Journal, Vol. XXIV, 1902) that in this case $\alpha + \beta \leqq \alpha' + \beta'$. Another form of $\star 2$ is $\star 1$, which is the most useful form for deductions. By the help of $\star 1$, it is proved ($\star 3$ and $\star 4$) that if n be a finite number and α_0 (as usual) the first infinite number, and if $n \times \beta = \alpha_0 \times \beta$, then $\beta = \alpha_0 \times \beta$. It is already known (cf. Zermelo, Gott. Nachricht., 1901) that if $\beta = n \times \beta$, then $\beta = \alpha_0 \times \beta$; and also it is obvious that if $\beta = \alpha_0 \times \beta$, then $\beta = n \times \beta$: then the present theorem completes this set of results.

Furthermore it is proved ($\star 6$) that if α is of the form $\alpha_0 \times \gamma$ [i. e. an $N\alpha_0$, cf. *Cardinal Numbers*, $\star 30$] and $\beta < \alpha$, then $\beta + \sigma = \alpha$ implies $\sigma = \alpha$. This proposition was assumed without proof in my article on *Cardinal Numbers* (cf. $\star 16 \cdot 22$, for example). It is already known that $\beta + \alpha = \alpha$.

It will be noticed that a condition attaches itself to the hypothesis of $\star 1$, and thence to the hypotheses of the succeeding theorems, namely (stating it generally), that any two numbers less than the numbers considered, are such that one of them is either greater than, or equal to, or less than the other. I am not aware of any proof that one of these relations must hold for any two cardinal numbers; but classes of infinite cardinal numbers are known for which this condition is true.

The proofs are written in Peano's notation explained in the article on *Cardinal Numbers*, already cited; except that here $\subset$ is used for "is contained in," as applied to classes, and $\supset$ is used for "implies," as applied to propositions, instead of $\supset$ for both these ideas. Also, in the proofs where a hypothesis is made to hold for the remainder of the proof "constr" for "constructive" is written after it.

★1 $\alpha, \beta, \alpha', \beta' \,\varepsilon\, Nc \,.\, \alpha + \beta = \alpha' + \beta' : \mu \leqq \alpha \,.\, \nu \leqq \beta \,.\supset_{\mu,\nu}.$
$\mu \lesseqgtr \nu : \supset : \alpha \geqq \alpha' \,.\vee.\, \beta \geqq \beta'$,

$[R \,\varepsilon\, 1 \rightarrow 1 \,.\, u \cap v = \Lambda \,.\, u \cup v = \rho \,.\, u' \cap v' = \Lambda \,.\, u' \cup v' = \breve{\rho}$, constr (1)

$S \varepsilon 1 \rightarrow 1 \,.\, \sigma \subset u \cap \rho v' \,.\, \breve{\sigma} = v \cap \rho u'$, constr (2)

$p = (u \cup v) - (\sigma \cup \breve{\sigma})$. constr (3)

$R' = R_p \cup SR \cup \breve{S}R$, constr (4)

$(4) \,.\supset.\, R' \,\varepsilon\, 1 \rightarrow 1 \,.\, \breve{\rho}' u = \breve{\rho}(u - \sigma) \cup \breve{\rho}(v \cap \rho u')$, (5)

$(5) \,.\supset.\, u' \subset \breve{\rho}' u \,.\supset.\, Nc\text{‘}u' \leqq Nc\text{‘}u$, (6)

$(2) \,.\, (6) \,.\supset: Nc\text{‘}(u \cap \rho v') \geqq Nc\text{‘}(v \cap \rho u') \,.\supset.\, Nc\text{‘}u' \leqq Nc\text{‘}u$. (7)

Similarly $Nc\text{‘}(v \cap \rho u') \geqq Nc\text{‘}(u \cap \rho v') \,.\supset.\, Nc\text{‘}v' \leqq Nc\text{‘}v$, (8)

$(7) \,.\, (8) : \mu \leqq \alpha \,.\, \nu \leqq \beta \,.\supset_{\mu,\nu}.\, \mu \lesseqgtr \nu : u \,\varepsilon\, \alpha \,.\, v \,\varepsilon\, \beta \,.\, u' \,\varepsilon\, \alpha' \,.\, v' \,\varepsilon\, \beta' : \supset :$ Prop],

★2 $\alpha, \beta, \alpha', \beta' \,\varepsilon\, Nc \,.\, \alpha < \alpha' \,.\, \beta < \beta' : \mu \leqq \alpha \,.\, \nu \leqq \beta \,.\supset_{\mu,\nu}.\, \mu \gtreqless \nu : \supset.$
$\alpha + \beta < \alpha' + \beta'$,

[★1 $.\supset.$ Prop.]

★3 $m, n \,\varepsilon\, Nc$ fin $- \iota 0 \,.\, m \geqq n \,.\, \beta \,\varepsilon\, Nc : \mu < \beta \,.\, \nu < \beta \,.\supset_{\mu,\nu} \mu \gtreqless \nu :$
$(m + n) \times \beta = \alpha_0 \times \beta : \supset .\, m \times \beta = \alpha_0 \times \beta$.

[★1 . Hyp $.\supset: m \times \beta + n \times \beta = \alpha_0 \times \beta + \alpha_0 \times \beta \,.\supset:$
$m \times \beta \geqq \alpha_0 \times \beta \,.\vee.\, n \times \beta \geqq \alpha_0 \times \beta$, (1)

$l \,\varepsilon\, Nc$ fin $.\, l \times \beta \geqq \alpha_0 \times \beta \,.\supset.\, l \times \beta = \alpha_0 \times \beta$, (2)

$m \geqq n \,.\supset: n \times \beta = \alpha_0 \times \beta \,.\supset.\, m \times \beta = \alpha_0 \times \beta$, (3)

(1) . (2) . (3) $.\supset.$ Prop].

★4 $n \,\varepsilon\, Nc$ fin $- \iota 0 \,.\, \beta \,\varepsilon\, Nc : \mu < \beta \,.\, \nu < \beta \,.\supset_{\mu,\nu}.\, \mu \gtreqless \nu :$
$n \times \beta = \alpha_0 \times \beta : \supset .\, \beta = \alpha_0 \times \beta$.

[★3 $.\supset.$ Prop.]

★5 $m, n \,\varepsilon\, Nc$ fin $- \iota 0 \,.\, m \sim n \,.\, \beta \,\varepsilon\, Nc : \mu < \beta \,.\, \nu < \beta \,.\supset_{\mu,\nu}.\, \mu \gtreqless \nu :$
$m \times \beta = n \times \beta : \supset .\, \beta = \alpha_0 \times \beta$,

$[m \times \beta = n \times \beta \,.\supset.\, m \times \beta = \alpha_0 \times \beta$. (1)

Note: This theorem (1) is given by Zermelo, Gott. Nachrichten, 1901.

(1) . ★4 $.\supset.$ Prop.]

★6 $\alpha \,\varepsilon\, N\alpha_0 : \mu \leqq \alpha \,.\, \nu \leqq \alpha \,.\supset_{\mu,\nu}.\, \mu \gtreqless \nu : \beta < \alpha \,.\, \beta + \sigma = \alpha : \supset : \sigma = \alpha$,

[Hyp $.\, \beta + \sigma = \alpha \,.\supset.\, \beta + \sigma = \alpha + \alpha$, (1)

(1) . ★1 $.\, \beta < \alpha \,.\supset.\, \sigma \geqq \alpha$, (2)

$\beta + \sigma = \alpha \,.\supset.\, \sigma \leqq \alpha$, (3)

(2) . (3) $.\supset.$ Prop.]

TRINITY COLLEGE, CAMBRIDGE, ENGLAND, *January*, 1903.

The Caustic, by Reflection, of a Circle.

BY T. J. I'A. BROMWICH.

Since the publication of Cayley's "Memoir upon Caustics,"* but little has been done in this direction. In the text-books of Geometrical Optics (e. g. Heath and Herman) Cayley's results and methods have been reproduced without alteration.

Cayley's work in the main is based upon the *point*-equation of the caustic (which is, of course, deduced from the equation of the reflected ray). But it is simpler, for most purposes, to work from the *line*-equation of the caustic,† or, what is really the same thing, to work with the coordinates of the reflected ray, expressed as algebraic polynomials of a parameter t.

Moreover, as in most questions dealing with circles and curves derived from circles, it is advantageous to use (instead of rectangular Cartesian coordinates ξ, η) the complex variables $x = \xi + i\eta$, $y = \xi - i\eta$. The use of x, y simplifies the arithmetic and makes most of the results more compact.‡

The following work is equivalent to that contained in Arts. VIII–XI XVII–XXV of Cayley's memoir. It appears that, for the problem of refraction (either at a line or at a circle) the method used below does not lead to any special simplification; and, accordingly, I have not reproduced any of my work on refracted caustics.

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* Phil. Trans. Roy. Soc. Lond., Vol. 147 (1857), p. 273 = Collected Math. Papers, Vol. 2, p. 336 (No. 145).

† That this is the more natural method seems clear from the singularities which are found; thus we have 6 cusps (at least 2 of which are real) and no inflexions.

‡ For illustrations of this, see Morley's papers in the American Journal of Mathematics (Vols. XIII, XVI) and in Transactions of the American Mathematical Society (Vols I, IV).

Incidentally, I have noted a few misprints in Cayley's memoir, which I have remarked in footnotes.

1. Take the reflecting circle as the unit circle

$$\xi^2+\eta^2=1, \text{ or } xy=1, \text{ if } x=\xi+i\eta,\ y=\xi-i\eta.$$

Then a point Q on the circle is given by the equations

$$x=t, \quad y=1/t,$$

where

$$t=e^{i\phi},$$

and ϕ is the angle between the radius OQ and the axis of ξ.

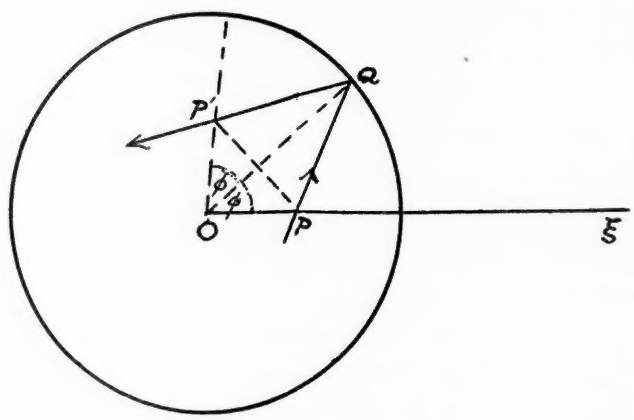
Suppose the rays start from a point P on the axis of ξ, say,

$$x=c, \quad y=c,$$

where, for convenience, c may be always supposed positive.

Then P', the reflection of P in OQ, is given by

$$OP'=OP=c,$$
$$P'OP=2\phi,$$



so that, for P', we have

$$x=ct^2, \quad y=c/t^2.$$

The reflected ray, corresponding to an incident ray PQ, is clearly QP', whose equation is

$$\begin{vmatrix} x, & y, & 1 \\ t, & 1/t, & 1 \\ ct^2, & c/t^2, & 1 \end{vmatrix}=0,$$

$$\begin{aligned} &\text{or} \qquad x(c-t)+yt^3(1-ct)+ct(t^2-1)=0, \\ &\text{i. e.,} \qquad -cyt^4+(c+y)t^3-(c+x)t+cx=0. \end{aligned} \tag{1}$$

Since the equation to the reflected ray is of degree 4 in t, the *class* of the caustic is also 4; and further, the caustic is unicursal. By differentiating the equation (1) with respect to t, and solving for x, y, we find

$$\frac{x}{ct^3(ct^3-3ct+2)}=\frac{y}{c(2t^3-3ct^2+c)}=\frac{1}{t^2[3ct^2-2(1+2c^2)t+3c]},$$

so that the caustic is of degree 6. These expressions are easily seen to be equivalent to those given by Cayley in Art. XIX of his memoir (in which, however, the denominators should be $1-3a\cos\theta+2a^2$ instead of $1-3a\cos 2\theta+2a^2$). These expressions for x, y will not be of very much service to us; but it will be useful to obtain explicitly the point-equation of the caustic. The t discriminant of the quartic (1) is

$$g_2^3-27g_3^2=0,$$

where

$$\begin{aligned} g_2 &= \tfrac{1}{4}(x+c)(y+c)-xyc^2=\tfrac{1}{4}[xy(1-4c^2)+c(x+y)+c^2], \\ g_3 &= -\tfrac{1}{16}c(x-y)(xy-c^2). \end{aligned}$$

hence the point-equation of the caustic is

$$-27c^2(x-y)^2(xy-c^2)^2+4[xy(1-4c^2)+c(x+y)+c^2]^3=0. \tag{2}$$

This investigation of (2) is not essentially distinct from the second one given by Cayley (Art. XVIII).

2. Since the cubic is unicursal and of degree 6, it will have 10 nodes $(=\frac{1}{2}\,\overline{6-1}\,.\,\overline{6-2})$, including cusps; and since it is of class 4, it has 3 bitangents $(=\frac{1}{2}\,\overline{4-1}\,.\,\overline{4-2})$, including inflexional tangents. Thus, using Plücker's equations, we see that—

The caustic has 4 nodes, 6 cusps, 3 bitangents and no inflexions.

The equation given by Klein (Math. Ann., Vol. X) relating to the reality of singularities, shows that the number of *real* cusps is

$$2(1+\tau''-\delta''),$$

where τ'' is the number of isolated real bitangents, and δ'' is the number of isolated real nodes (acnodes).

3. *To determine the cusps*, we make the equation (1), regarded as an equation in t, have three equal roots α, α, α; the fourth root is then* $-\alpha$ (unless the repeated root is $t=0$ or $t=\infty$, cases which are to be discussed separately). In this way we find the relations

$$\frac{1}{x}+\frac{1}{c}=\frac{2}{\alpha},\quad \frac{1}{y}+\frac{1}{c}=2\alpha,\quad \frac{x}{y}=\alpha^4,$$

leading to
$$\alpha^4-2\alpha^3c+2\alpha c-1=0.$$

Thus we have

$$\text{(i).}\quad \alpha=\pm 1,\quad x=y=-c/(1\pm 2c),$$

two cusps which are always *real;* and

$$\text{(ii).}\quad \alpha^2-2\alpha c+1=0,\quad x=c\alpha^2,\quad y=c/\alpha^2,$$

two cusps which are real if $|\alpha|=1$, or if $c<1$; here the line joining the cusps is

$$\xi=\tfrac{1}{2}(x+y)=c(2c^2-1),$$

and they are on the circle $xy=c^2$, which passes through the bright point.

If $t=0$ is a triple root of (1), we have

$$\frac{x}{0}=\frac{y}{1}=\frac{c}{0},$$

giving one of the circular points as a cusp; in like manner, if $t=\infty$ is a triple root, we find the other circular point as a cusp.

Hence we have now 6 cusps, agreeing with the number determined in §2; and, by examining equation (2), we verify the correctness of the work. The number of *real* cusps is 2 if $c>1$, or 4 if $c<1$. Thus,

$$\tau''-\delta''=0,\quad c>1,$$
$$\tau''-\delta''=1,\quad c<1.$$

4. *The bitangents.*

Construct first the relations between t_1, t_2, t_3, t_4 the parameters of the points of contact of the tangents drawn from any point to the caustic. From equation (1) we find

$$c(\Sigma t_1+\Sigma t_1t_2t_3)=1+t_1t_2t_3t_4,\quad \Sigma t_1t_2=0.$$

---

* This follows at once from the fact that equation (1) contains no terms in t^2.

Thus if $p = t_1 + t_2, \quad q = t_1 t_2, \quad r = t_3 + t_4, \quad s = t_3 t_4,$

we have $c(p + r + qr + ps) = 1 + qs, \quad q + s + pr = 0.$

From these equations r, s can, in general, be found in terms of p, q; but if t_1, t_2 belong to a bitangent, it is clear that t_3, t_4 (and so r, s) must be indeterminate. Thus for the bitangents, we have the conditions

$$(cp - 1) - q(cp - q) = 0, \quad c(1 + q) - p(cp - q) = 0.$$

The first of these conditions is

$$(1 - q)(cp - q - 1) = 0,$$

so we have two cases,

(i). $q = 1$, giving $cp^2 - p - 2c = 0$, i. e., $p = \dfrac{1}{2c}\left[1 \pm (1 + 8c^2)^{\frac{1}{2}}\right]$,

(ii). $cp = q + 1$, giving $p = 0, \quad q = -1$.

Remembering that the bitangent is a tangent at $t = t_1$, $t = t_2$, we deduce that its equation must be

$$x + y\left[q - p^2 + cp(p^2 - 2q)\right] = c(p^2 - q - 1).$$

Thus, in case (i), the bitangent is $x + y = p$ or $\xi = \frac{1}{2}p$; that is,

$$\xi = \frac{1}{4c}\left[1 \pm (1 + 8c^2)^{\frac{1}{2}}\right],$$

while, in case (ii), the bitangent is $x - y = 0$ or $\eta = 0$. We have, accordingly, found the three bitangents; in case (i), the points of contact will be real if $|p| < 2$, which is always satisfied by the line $\xi = \dfrac{1}{4c}\left[1 - (1 + 8c^2)^{\frac{1}{2}}\right]$, but is only satisfied by the other bitangent if $c > 1$. In case (ii), the points of contact are always real, namely, the two cusps on the line $\eta = 0$.

Hence, from §§2, 3, we see that

$$c > 1, \quad \tau'' = 0, \quad \delta'' = 0,$$
$$c < 1, \quad \tau'' = 1, \quad \delta'' = 0.$$

The equations just found for the bitangents agree with those given by Cayley in Art. XXI of his memoir; Cayley finds, however, the tangents which are parallel to the axis of η, and these are clearly bitangents, owing to symmetry. From this point of view, we are to have equation (1) of the form

$$x + y = \text{const.},$$

so that $$c - t - t^3 + ct^4 = 0,$$

which may be written in the form

$$c(t+1/t)^2-(t+1/t)-2c=0.$$

In this way we get back to the original forms.

So also we can find tangents parallel to the axis of ξ, from the condition

$$c-t+t^3-ct^4=0,$$

i. e.,

$$(1-t^2)[c(1+t^2)-t]=0.$$

The solution $t=\pm 1$ gives the third bitangent $\eta=0$; and the solution $c(t^2+1)=t$ gives the pair of parallel tangents

$$x-y=t-1/t=\pm i(4c^2-1)^{\frac{1}{2}}/c$$

or

$$2c\eta=\pm(4c^2-1)^{\frac{1}{2}},$$

as given by Cayley.

5. *The nodes.*

If (x, y) is a node of the caustic, the equation (1) is a perfect square, and so we can obtain the corresponding values of t (t_1, t_2, say) by writing $r=p$, $s=q$ in the first set of equations of §4. Thus

$$2cp(1+q)=1+q^2, \quad 2q+p^2=0,$$

and the corresponding values of x, y are given by

$$x=-q^2y, \quad y+c=2pcy.$$

Hence

$$x=qc(1+q)/(1-q), \quad y=c(q+1)/q(q-1),$$

and q is a root of the quartic

$$(1+q^2)^2+8c^2q(1+q)^2=0.$$

Thus there are four nodes, which are all imaginary; this can be proved by finding ξ, η. We have

$$\xi=\tfrac{1}{2}(x+y)=-\tfrac{1}{2}c(1+q)^2/q=-c(k+1)$$

if

$$k=\tfrac{1}{2}(q+1/q).$$

Now k satisfies

$$k^2+4c^2(k+1)=0,$$

and so

$$k=2c[-c\pm(c^2-1)^{\frac{1}{2}}],$$

thus

$$\xi=k^2/4c=c[-c\pm(c^2-1)^{\frac{1}{2}}]^2.$$

Also

$$\eta=(x-y)/2i=-ikc(1+q)/(1-q)$$

and

$$\left(\frac{1+q}{1-q}\right)^2=\frac{k+1}{k-1}=\frac{c\mp(c^2-1)^{\frac{1}{2}}}{3c\pm(c^2-1)^{\frac{1}{2}}},$$

since
$$k = 2c/[-c \mp (c^2 - 1)^{\frac{1}{2}}].$$

These values of ξ, η agree with Cayley's, given in Art. XXII of his memoir; but it is possible to verify that the nodes are imaginary, without carrying out the calculations quite so far. In fact, if (x, y) is a real point, we must have
$$|x| = |y| \quad \text{or} \quad |q| = 1.$$

Thus k must be real and lie between $+1$ and -1; but it is clearly impossible to satisfy the equation
$$k^2 + 4c^2(k+1) = 0$$

or real values of k, which are greater than -1. That is, (x, y) must be an imaginary point.

Since *all* the nodes have been proved to be imaginary, it follows that there are no real acnodes; which agrees with §4.

6. *The asymptotes.*

If we make two consecutive tangents parallel,* we find the condition
$$t^2\,[3c(t^2+1) - 2(1+2c^2)\,t] = 0$$

and rejecting the factor $t = 0$, which corresponds to the cuspidal tangent at one of the circular points, we have that the values of t giving the asymptotes are given by
$$3c(t^2+1) - 2(1+2c^2)\,t = 0,$$
or by
$$3ct = 1 + 2c^2 \pm i\,(4c^2-1)^{\frac{1}{2}}(1-c^2)^{\frac{1}{2}}.$$

The asymptotes are real if $|t| = 1$, or if $1 > c > \frac{1}{2}$; and then the explicit equations are given by substituting these values of t in equation (1). Thus, for instance, the perpendicular from the origin† on an asymptote is
$$\pm\,\tfrac{1}{2}\,\frac{ct\,(t^2-1)}{[(t^3-ct^4)(c-t)]^{\frac{1}{2}}} = \pm\,\frac{\frac{1}{2}c\,(t-1/t)}{[c\,(t+1/t)-(1+c^2)]^{\frac{1}{2}}} = [\tfrac{1}{3}(4c^2-1)]^{\frac{1}{2}}.$$

---

* An alternative method is to make x, y infinite when expressed in terms of t. See the expression given in §1.

† The perpendicular from the origin on the line $lx + my + n = 0$ is $\pm\frac{1}{2}\,n/(lm)^{\frac{1}{2}}$.

The final equations of the asymptotes are

$$\eta = \pm \frac{(4c^2-1)^{\frac{3}{2}}}{(1-c^2)^{\frac{1}{2}}(1+8c^2)}\left(\xi - \frac{3c}{4c^2-1}\right),$$

as found by Cayley (Art. XXI of his memoir).

7. *The caustic cuts the reflecting circle* in the points given by $xy = 1$, taken together with equation (2). Expressing everything in terms of $(x+y) = 2\xi$, we find

$$(c\xi - 1)^2[c\xi - 1 + \tfrac{9}{8}(1-c^2)(1+3c^2)] = 0,$$

so that, if $c > 1$, the caustic touches the circle at the two real points given by $c\xi = 1$; if $c < 1$, this contact is imaginary.

The points given by $c\xi = 1 - \frac{9}{8}(1-c^2)(1+3c^2)$ are imaginary if $c > 1$; for we have*

$$c^2(\xi^2 - 1) = \tfrac{1}{64}(1-c^2)(1-9c^2)^3;$$

and so these points are real only if $1 > c > \frac{1}{3}$. Summing up, we see that

$c > 1$, the caustic has double contact with the circle in real points ($\xi = 1/c$).

$1 > c > \frac{1}{3}$, the caustic cuts the circle in two real points,

$$c\xi = 1 - \tfrac{9}{8}(1-c^2)(1+3c^2).$$

$\frac{1}{3} > c$, the caustic is within the circle.

8. *The foci of the caustic.*

If the tangent, as given by equation (1), passes through the circular point

$$\frac{x}{0} = \frac{y}{1} = \frac{1}{0}$$

we must have

$$t^3(1 - ct) = 0.$$

The value $t = 0$ gives the tangent $x = 0$, which is the cuspidal tangent at this circular point; and $t = 1/c$ gives the tangent $x = 1/c$.

Hence the four tangents from this circular-point are included in the equation

$$x^3(cx - 1) = 0.$$

---

* Cayley's arithmetic appears to be a little wrong here. For he states that $\xi = \pm 1$ gives $(c \pm 1)(27c^2 + 9c + 1) = 0$, and he deduces the condition for reality $c < 1$, without including $c > \frac{1}{3}$ (Art. XXI). But later on (Art. XXV) he states the correct result for $c < \frac{1}{3}$.

Similarly, the four tangents from the other circular point are determined by

$$y^3(cy-1)=0.$$

It follows that $x=0$, $y=0$ is a triple focus, and that the only other real focus is $x=1/c$, $y=1/c$.

To make up the total of 16 foci belonging to curve of class 4, it should be observed that the triple focus counts for 9; and that there are two more imaginary foci ($x=1/c$, $y=0$ and $x=0$, $y=1/c$) each counting for 3; then $9+6+1=16$

Hence:

The real foci of the caustic consist of (i) *a triple focus at the centre of the reflecting circle; and* (ii), *a single focus at the point inverse to the bright point with respect to the reflecting circle.*

9. We have now all the *general* theorems for the caustic; but the special cases, $c=1$, ∞, deserve a little consideration. Take first the case $c=1$, when *the origin of light is on the reflecting circle.* The equation (1) becomes

$$(t-1)[x+yt^3-t(t+1)]=0$$

and equation (2) becomes

$$(x+y-2)^2[27x^2y^2-18xy-4(x+y)-1]=0.$$

Thus the curve degenerates into the line $x+y-2$ counted twice and the cardioid

$$27x^2y^2-18xy-4(x+y)-1=0,$$

as given by Cayley in Art. XXIII of his memoir. Starting from the tangential equation

$$x+yt^3-t(t+1)=0$$

we find that points on the caustic are given by*

$$3x=2t+t^2,\quad 3y=2/t+1/t^2,$$

an epicycloid (cardioid) given by the rolling of a circle of radius $\frac{1}{3}$ on an equal circle. Similarly, the line equation gives, for the degenerate form, the point $x=y=1$ and the same cardioid.

In this case, the class is 3 and the degree is 4; there is one bitangent, $\xi=-\frac{1}{2}$; there are 3 cusps (at $\xi=-\frac{1}{3}$, $\eta=0$ and at the two circular points);

* These values are also found by putting $c=1$ in the expressions given in §1, for x, y in terms of t.

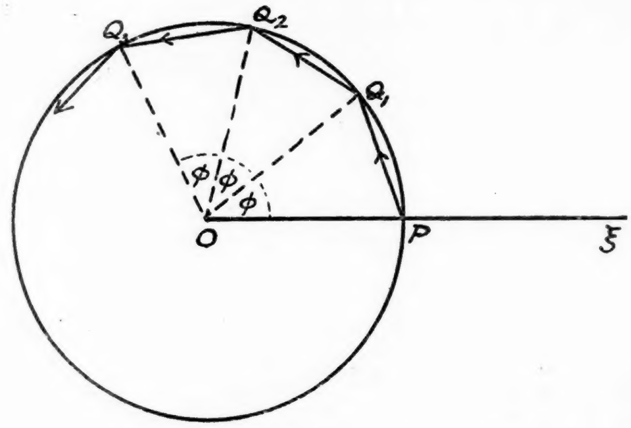
the imaginary nodes disappear ;* there are no real asymptotes and there are no inflexions (as in the general caustic). The point of contact of the cardioid and line $\xi = 1$ counts as the other 3 cusps of the general case ; while $\xi = 1$ is the limiting form of a bitangent.

If $c = \infty$, so that *the incident rays are parallel to* $o\xi$, we find that equation (1) gives

$$x - yt^4 + t(t^2 - 1) = 0$$

leading to $$4x = 3t - t^3, \quad 4y = 3/t - 1/t^3$$

for points on the caustic. This proves that the caustic is an epicycloid of class 4 and of degree 6. This curve has the two real cusps $\xi = \pm \frac{1}{2}, \eta = 0$; and the two imaginary nodes $\xi = 0, \eta = \pm 2i/\sqrt{3}$, the remaining singularities are all at the two circular points, there being two cusps and a node at each, as is easily verified by taking the penultimate form. The three bitangents are $\eta = 0$, $\xi = \pm 1/\sqrt{2}$.



This special caustic can be obtained by rolling a circle of radius $\frac{1}{4}$ upon a circle of radius $\frac{1}{2}$. The point equation is

$$4(4xy - 1)^3 + 27(x - y)^2 = 0,$$

as given by Cayley (Art. XXIV).

10. In the two cases mentioned in §9, the caustic can be easily found after

* If the cardioid and line are taken together, the 4 nodes coincide in pairs at the two imaginary intersections of the line and the cardioid. That is, at the points

$$\xi = 1, \quad \eta = \pm 2i/\sqrt{3}.$$

n reflections. Taking first the case when the origin of light is on the reflecting circle, it is easily seen that the n^{th} point of incidence Q_n is

$$x = t^n, \quad y = 1/t^n.$$

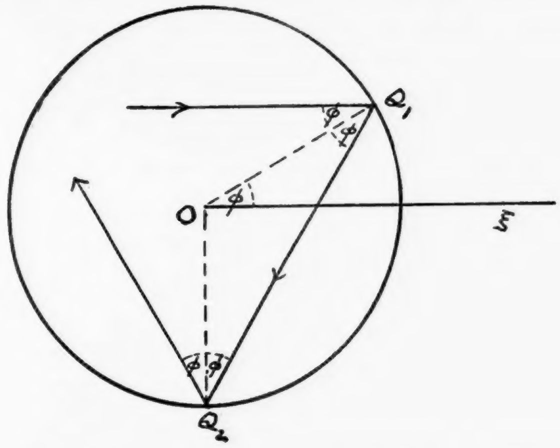
Hence the equations to the ray, after n reflections, is

$$x + yt^{2n+1} = t^n(t+1),$$

since this is the equation to the line $Q_n Q_{n+1}$. If we write $n = 1$, we get back to the equation of §8. We deduce at once, by differentiating with respect to t, the coordinates of a point on the envelope in the form

$$(2n+1)x = (n+1)t^n + nt^{n+1},$$
$$(2n+1)y = (n+1)/t^n + n/t^{n+1}.$$

These equations show that the envelope is an epicycloid of class $(2n+1)$ and of degree $2(n+1)$. It may be found geometrically by rolling a circle of radius



$n/(2n+1)$ upon one of radius $1/(2n+1)$; agreeing with what was found for the case $n = 1$.

The determination of the singularities need not be undertaken here, as those of the epicycloids are well known.

For the case when the incident rays are parallel to ξ, we see that Q_{n+1} is the point

$$x = (-1)^n t^{2n+1}, \quad y = (-1)^n/t^{2n+1}.$$

Thus the n^{th} reflected ray (since it joins Q_{n+1} to Q_n) is

$$x - yt^{4n} = (-1)^n t^{2n-1}(t^2 - 1)$$

leading to the caustic

$$4nx = (-1)^n [(2n-1)\,t^{2n+1} - (2n+1)\,t^{2n-1}],$$
$$4ny = (-1)^n [(2n-1)/t^{2n+1} - (2n+1)/t^{2n-1}].$$

This is an epicycloid of class $4n$ and of degree $(4n+2)$. It may be obtained by rolling a circle of radius $(2n-1)/4n$ upon one of radius $1/2n$. In both these cases the epicycloids have a finite number of *real* nodes.

These cases are given by Cayley (Arts. X, XI).

QUEEN'S COLLEGE, GALWAY.

On Imprimitive Substitution Groups.

By Harry Waldo Kuhn.

Introduction.

The study of substitution groups first arose in connection with the solution of algebraic equations. In the earliest work that devotes considerable attention to these groups (Ruffini, Teoria generale delle equazioni, Bologna, 1799), we find the non-cyclic groups divided into three classes which correspond to intransitive, primitive and imprimitive groups. Of the substitution groups considered in this work, the group of the order 8 and degree 4 is the only imprimitive group that receives any attention. In a memoir two years later,* Ruffini shows that the group of an irreducible equation is transitive. When the group of an equation in x is not primitive, Jordan has proved† that the equation is the result of the elimination of y from two irreducible equations of the form

$$\begin{aligned} y^m + a_1 y^{m-1} &+ \dots + a_m &= 0, \\ x^n + b_1(y)\, x^{n-1} &+ \dots + b_n(y) &= 0, \end{aligned}$$

and conversely.

The systems of imprimitivity of any imprimitive group G are permuted by its substitutions according to a transitive group P that has a $1, \alpha$ isomorphism to G, and whose degree equals the number of systems in the given set. The invariant subgroup of G that corresponds to identity in P is intransitive and its substitutions leave the given systems unchanged.‡ It is called the head of G and its order may equal unity.|| If the group P is itself imprimitive, the corresponding systems can be united into larger ones which are permuted by the substitu-

* Memoire della società italiana delle scienze, Vol. 9, pp. 144–526. Modena, 1801.

† Traité des Substitutions, p. 259.

‡ Jordan, loc. cit., p. 41; ibid., p. 399.

|| Dyck, Mathematische Annalen, Vol. 22 (1883), pp. 94, 108; cf. also Miller, Bulletin of American Mathematical Society, Vol. 1 (1894), p. 257.

tions of G according to a primitive group.* The degree of any solvable primitive group is the power of a prime.† In order that an equation can be solved by radicals, its group must be solvable. It follows directly that any imprimitive group that belongs to a solvable equation has at least one set of p^n systems of imprimitivity when p is a prime number.

If a given imprimitive group contains two distinct sets of systems of imprimitivity, then under certain conditions new sets of systems can be formed from these. This can always be done in case some system of the one set has more than one element in common with some system of the other set.‡ It is not true in general, however, that a new set of systems can be formed by combining all the systems of one set that have any elements in common with a given system of the other set. In his Traité des Substitutions, p. 34, Jordan states a theorem that says this can be done, but he afterwards notes the error himself.|| Starting with this theorem he proves some very interesting results in reference to what he terms "Facteurs de Non-Primitivitat." An interesting problem presents itself here in the discussion of the imprimitive groups for which the theorem is true. An important property of such groups has recently been proved by Maillet.§

The important problem of determining when a given group can be represented as a transitive group of a given degree (or in particular as an imprimitive group) has been completely solved by Dyck.¶ When the properties of the given group are known his investigations give all the ways in which such a representation can take place. They do not determine, however, how many of the different representations of the same group are distinct as substitution groups.** This question finds its answer in a theorem due to Miller.†† In the particular case when the degree equals the order there is just one such group.

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* Jordan, loc. cit., p. 399.

† Galois, Oeuvres Mathematiques, p. 27. Cf. also Jordan, l. c., p. 398.

‡ Jordan, loc. cit., p. 34.

|| Giornale di Matematiche, Vol. 10 (1872), p. 116.

§ Bulletin de la Societé Mathématique de France, Vol. 28 (1900), p. 15.

¶ Mathematische Annalen, Vol. 22 (1883), p. 94.

** Burnside, Messenger of Mathematics, Vol. 23 (1898), p. 103.

†† Bulletin of the American Mathematical Society, 2d Series, Vol. 3 (1896), p. 215. Cf. also Giornale di Matematiche, Vol. 38 (1900), pp. 1-9.

The investigations of Dyck just referred to determine also the different sets of systems of imprimitivity which are admitted by a given imprimitive group. In particular, when the group is regular, the number of such sets is shown to be equal to the number of subgroups (not counting identity) that are contained in the group. It is clear that any substitution that is commutative with all the substitutions of a given imprimitive group determines systems of imprimitivity of the group. The number of such substitutions for any regular group is well known to be equal to the order of the group.*

The problem that has received the most attention recently in the study of imprimitive groups relates to the construction of such groups. The enumeration of the imprimitive groups of a given degree has been carried through degree fourteen. The methods used in forming these lists have been chiefly of a tentative nature. Recently, however, some theorems have been established that are useful in the determination of imprimitive groups of certain kinds.†

Any regular group of composite order is imprimitive. The determination of the number of distinct groups of a given order has been studied from the point of view of abstract groups and from that of substitution groups. By means of the latter method the regular groups whose order is less than 48 have been constructed.‡

Some imprimitive groups which do not belong to either of the two classes just mentioned have also been enumerated. These include certain groups whose orders are of a particular form. The orders that have been considered are 1) $p \cdot q \cdot \gamma$‖; 2) p_3§; and 3) $8p$¶, when p, q and γ are distinct prime numbers. Another important class of transitive groups that has been studied is formed by the groups which are isomorphic to the symmetric and the alternating groups of a given degree.** The necessary and sufficient condition that a group is multiply isomorphic to a non-regular transitive group has also been determined.††

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* Jordan, Journal de l'École Polytechnique, Vol. 22 (1861), p. 153

† For full references to those through degree 10 cf. Miller, Bulletin of the American Mathematical Society, Vol. 2 (1895), pp. 138–145. Those of degree 12 and 14 are determined by Miller, Quarterly Journal of Mathematics, Vol. 28 (1895), p. 193, and Vol. 29 (1897), p. 234. Cf. also American Journal of Mathematics, Vol. 21 (1899), p. 287.

‡ Ibid., Quarterly Journal of Mathematics, Vol. 28 (1895), p. 232.

‖ Ibid., Bulletin of the American Mathematical Society, Vol. 2 (1886), pp. 213–222.

§ Ibid., Annals of Mathematics, Vol. 10 (1896), pp. 156–8.

¶ Ibid., Philosophical Magazine (5), 43 (1896), pp. 117–125; cf. Cayley.

** Maillet, Journal de Mathématiques, 5 série, Vol. 1 (1895), pp. 5–34.

†† Miller, Giornale di Matematiche, Vol. 38 (1900), p. 8.

In the preparation of the following paper I am indebted to Dr. Miller for helpful suggestions and criticisms.

The first section of the paper relates to the imprimitive groups whose elements can be divided into systems of imprimitivity in more than one way and whose substitutions permute all the sets of systems according to primitive groups. A few properties of the heads of such groups are first given. These are followed by the study of the groups that contain a given number of heads. Those that contain more than two heads, all different from identity, receive the most attention. The cases for which one or more of the heads reduces to identity are then considered. A theorem is also given that relates to the holomorph of an abelian group of order p^m and type $(1, 1, \ldots, 1)$.

The second section considers the substitutions which are commutative with each substitution of a given transitive group. Jordan's theorem on the number of substitutions that are commutative with each substitution of any regular group is generalized so as to apply to any transitive group.

Section III relates to the construction of the imprimitive groups whose substitutions permute the systems of intransitivity of the heads according to the metacyclic group of degree p or to one of its transitive subgroups of degree p. The heads considered are: 1), those whose transitive constituents are the symmetric or the alternating groups of degree $n\,(n > 2)$, and 2), those whose constituents are transitive subgroups of degree q having a given index under metacyclic groups of the same degree.

In section IV the results of section III are made use of to determine the imprimitive groups of degree fifteen.

SECTION I.—*On the imprimitive groups whose substitutions permute all their sets of systems of imprimitivity according to primitive groups.*

1. Let G denote an imprimitive group that has more than one set of systems of imprimitivity, and let the corresponding heads of G be denoted by H_1, H_2, etc. Suppose, further, that the systems that correspond to the head H_i are permuted by the substitutions of G according to the group P_i, where i equals $1, 2, \ldots$

THEOREM.—*If the heads H_1, H_2, ... are all different from identity, and if the groups P_1, P_2, ... are primitive, then*

(a). *The heads can have no substitutions in common besides identity, and hence*

(b). *Each substitution of H_i is commutative with each substitution of H_j (i and j being any two of the subscripts of the H's).*

(c). *Each head contains at least one substitution whose degree equals the degree of G.*

(d). *Any head H_i is formed by establishing a one-to-one isomorphism between its transitive constituents.*

(a). The systems of intransitivity of any head of an imprimitive group are systems of imprimitivity of this group, and they are permuted by its substitutions according to a fixed transitive group. It follows that the systems of imprimitivity of the groups we are considering must be the systems of intransitivity of the heads.* Consider now any two of the heads, H_1, H_2 say. It results directly from what has been stated that H_1 and H_2 cannot consist of the same substitutions.

Let us assume then that H_1 is contained in H_2. In this case there must be an α, 1 isomorphism between P_1 and P_2. The elements of any transitive constituent of H_2 are composed of the elements in a definite number (m say) of the transitive constituents of H_1. Let the two sets of systems of imprimitivity be denoted by

$$a_1, a_2, \ldots, a_m;\quad b_1, b_2, \ldots, b_m;\quad c_1, c_2, \ldots, c_m;\quad \ldots$$

and

$$a\quad,\quad b\quad,\quad c\quad,\ \ldots$$

those in the first row composing the elements of P_1 and those in the second row the elements of P_2. The subgroup h_α of order α in P_1 that corresponds to identity in P_2 can only permute the a's among each other, the b's among each other, etc. That is, h_α is intransitive, and hence P_1 is imprimitive.

Assume next that H_1 and H_2 have a common subgroup H_{12}. This group, H_{12}, is intransitive, and since it is contained in both H_1 and H_2, it is invariant in G. Its systems of intransitivity are then systems of imprimitivity. These systems, which are different from those of H_1 or H_2, are permuted by the substitutions of G according to some transitive group P. As H_{12} is contained in H_1 and H_2, the preceding argument shows that P is imprimitive. Hence, our proof of (a) is complete.

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* Miller, American Journal of Mathematics, Vol. 21 (1899), p. 305.

(b). That each substitution of H_i is commutative with each substitution of H_j (i and j being any two of the subscripts of the H's) follows at once from the theorem: If every operator of a group G_1 transforms the group G_2 into itself, and every operator of G_2 transforms G_1 into itself, then when G_1 and G_2 have only identity in common every operator of G_1 is commutative with every operator of G_2.*

(c). Let g, h_i and p_i be the orders of G, H_i and P_i respectively; let, further

$$1,\ S_2,\ S_3,\ \ldots\ldots,\ S_g$$

be the substitutions of G, and

$$1,\ S_2,\ S_3,\ \ldots\ldots,\ S_{h_1}$$

be those of H_1. Form the rectangular array

$$\begin{array}{llll} 1, & S_2, & \ldots\ldots, & S_{h_1}, \\ S_{h_1+1}, & S_2 S_{h_1+1}, & \ldots\ldots, & S_{h_1} S_{h_1+1}, \\ \vdots & \vdots & \ldots\ldots, & \vdots \end{array}$$

from the substitutions of G. None of the rows in this array can contain more than one substitution that belongs to any head different from H_1. For if there were two substitutions in any row that belong to H_2 (say), then the inverse of one of them multiplied by the other would give a substitution, different from identity, that belongs to both H_1 and H_2. This, however, cannot be true from what has just been proved. Further, to each row there corresponds one substitution of the group P_1. Hence, it follows that any head that differs from H_1 is simply isomorphic either to P_1 or to some invariant subgroup of P_1. Now an invariant subgroup of a primitive group is transitive, and every transitive group contains substitutions whose degree equals the degree of the group. Hence every head different from H_1 contains substitutions whose degree equals the degree of G. Similarly, by writing the substitutions of G in rectangular array with respect to the head H_2, we see that H_1 contains substitutions whose degree equals that of G.

(d). We have seen that any head, H_2, that is different from H_1, is simply isomorphic to P_1 or to some invariant subgroup of P_1. Denote by Q_1 that subgroup of P_1 to which H_2 is simply isomorphic. We shall prove now that the transitive constituents of H_2 are simply isomorphic to Q_1. To establish this it

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* Dyck, Mathematische Annalen, Vol. 22 (1883), p. 97.

is sufficient to prove that each substitution in H_2 involves elements from each of its transitive constituents. The subgroup Q_1 is transitive and so contains a substitution that puts any element into any other element. That is, H_2 contains a substitution that puts any system of H_1 into any other system. Since each substitution of H_1 is commutative with each substitution of H_2, it follows therefore that each substitution of H_1 contains elements from each of its transitive constituents. Similarly, each substitution of H_2 contains elements from each of its transitive constituents, and hence each constituent of H_2 is simply isomorphic to Q_1.

2. Let h_i denote the order of the head H_i.

THEOREM.—*If the order of G is equal to $h_1 h_2$, and if H_1 and H_2 are the only heads that differ from identity, then when $P_1, P_2, \ldots$ are primitive groups, G has just two sets of systems of imprimitivity.*

From the argument used in proving the theorem in paragraph 1, it is clear that H_1 and H_2 can have no substitutions in common besides identity, and hence that every substitution of H_1 is commutative with every substitution of H_2. It follows that G is the direct product of H_1 and H_2. We are to prove that G can have no set of systems of imprimitivity that are interchanged by its substitutions according to a simply isomorphic primitive group.

If G has such a set of systems, it must be possible to represent it as a primitive group. It follows therefore that H_1 and H_2 must be simply isomorphic simple groups of composite order and that G, when so represented, is of degree h_1.* The subgroup G_1 of G that gives rise to its representation in the primitive form is formed by establishing a simple isomorphism between H_1 and H_2. Let the substitutions of G be denoted by the symbols

$$1,\ S_2,\ S_3,\ \ldots\ ,\ S_{h_1^2}$$

and let them be written in rectangular array with respect to the substitutions of G_1. If the first h_1 of the above substitutions form the subgroup G_1, we have the arrangement

$$\begin{array}{lllll} 1, & S_2 & , \ldots, & S_{h_1}, & A_1, \\ S_{h_1+1}, & S_2 S_{h_1+1}, & \ldots, & S_{h_1} S_{h_1+1}, & A_2, \\ & & \ldots & & \\ & & \ldots & & \\ S_{2h_1-1}, & S_2 S_{2h_1-1}, & \ldots, & S_{h_1} S_{2h_1-1}, & A_{h_1}. \end{array}$$

* Burnside, Theory of Groups of Finite Order, p. 190; Miller, Transactions of the American Mathematical Society, Vol. 1 (1900), p. 70.

Denote the i^{th} row of this array by A_i where $i = 1, 2, \ldots, h_1$. The symbols A may be taken for the elements of G when represented in the primitive form. Also to each element A_i there corresponds a certain number of the elements of G. That is, the subgroup G_2 of G that gives rise to its representation in the given imprimitive form must be some subgroup of G_1. The subgroup G_2 must in fact be maximal in G_1; otherwise G would contain a set of systems of imprimitivity that is permuted by its substitutions according to an imprimitive group. Since G_2 is formed by establishing a simple isomorphism between two subgroups of H_1 and H_2, it follows that G_2 is contained in a subgroup M_1 whose order is the square of its order. It is also contained in a subgroup M_2 whose order is h_1 times its order and which contains M_1. It follows that one of the corresponding sets of systems is permuted by the substitutions of G according to an imprimitive group. Hence G has just two sets of systems of imprimitivity.

3. THEOREM.— *When G is regular and has just two heads that differ from identity, then if $P_1, P_2, \ldots$ are primitive groups, G is the cyclic group of order pq where p and q are distinct primes.*

The number of sets of systems that belongs to any regular group is equal to the number of its subgroups, not including identity or the whole group.* Hence the two heads that differ from identity must be generated by substitutions of prime order and the order of one must be different from that of the other.

4. THEOREM.—*If G contains more than two sets of systems of imprimitivity, and if $H_1, H_2, \ldots$ are all different from identity, then when $P_1, P_2, \ldots$ are primitive groups,*

(a). *The degree of any substitution besides identity of each head is equal to the degree of G.*

(b). *The heads are simply isomorphic abelian groups; each is of degree p^{2m} of order p^m and of type $(1, 1, \ldots, 1)$ where p is a prime and m is a positive integer.*

(a). We prove first that in any such group the heads can contain, besides

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* Dyck, Mathematische Annalen, Vol. 22 (1883), p. 89.

identity, only substitutions whose degree equals that of G. As in the theorem of paragraph 1, write the substitutions of G in rectangular array with respect to the substitutions of H_1. We know that any other head is simply isomorphic to P_1 or to some invariant subgroup of P_1. Consider the two heads H_2 and H_3 and let Q_2 and Q_3 denote the respective subgroups of P_1 to which these heads are simply isomorphic. Both Q_2 and Q_3 contain substitutions whose degree equals the degree of P_1, since an invariant subgroup of a primitive group is transitive. Further, neither Q_2 nor Q_3 can contain a substitution whose degree is less than that of P_1. This may be seen as follows: Each substitution of Q_2 is commutative with each substitution of Q_3, since the heads of H_2 and H_3 have this property. Suppose now that Q_3 contains a substitution S whose degree is less than that of P_1. Then the group $\{Q_2, S\}$ that is generated by Q_2 and S is transitive since Q_2 is transitive. As the substitution S is commutative with each substitution of Q, this transitive group will contain an invariant substitution S whose degree is less than the degree of the group. This, however, cannot be true, since an invariant substitution of a transitive group of degree n must be regular and of degree n. It follows, therefore, that the degree of each substitution, besides identity of any head that differs from H_1, is the same as the degree of G. By writing the substitutions of G in rectangular array with respect to H_2, it follows by a similar argument that H_1 also possesses this property.

(b). The group generated by any two of the heads, H_1 and H_2 say, must be transitive. If it were intransitive it would form a new head that contains both H_1 and H_2, and this cannot be true according to theorem 1 of this section. Further, this group $\{H_1, H_2\}$ must contain the substitutions of all the heads of G. For if it did not contain a substitution S of some other head, then $\{H_1, H_2\}$ and S would generate a transitive group whose substitutions all have the same degree as that of the group and which contains a number of substitutions that is greater than this degree. This, however, cannot be true. It follows, therefore, that the heads are simply isomorphic to each other.

In the group that is generated by the substitutions of H_2 and H_3 are found the substitutions of H_1. Also any substitution of H_2 or of H_3 is commutative with each substitution of H_1, and, therefore, any substitution in the group $\{H_2, H_3\}$ is commutative with each substitution of H_1. It follows that H_1, and hence also each of the heads of G, is abelian. Further, each head must be of

order p^m (where p is a prime and m is a positive integer) and of type $(1, 1, \ldots, 1)$. For if the substitutions of the heads were not all of the same prime order, then the subgroup of $\{H_1, H_2\}$ that is generated by its substitutions of lowest order would form a head that would not satisfy the requirements of the theorem in paragraph 1. Finally, since the group $\{H_1, H_2\}$ is regular, it follows that the degree of G is p^{2m}.

Corollary: *If G is regular it must be the non-cyclic group of order p^2.*

When G is regular it must coincide with the group generated by any two of its heads. It is then an abelian group of order p^{2m} and of type $(1, 1, \ldots, 1)$. The only groups of this type that satisfy the requirements of the above theorem are clearly the groups of order p^2.

5. Let P denote an abelian group of order p^m and of type $(1, 1, \ldots, 1)$ when represented as a transitive group in the elements

$$a_1, a_2, \ldots, a_{p^m}.$$

With each substitution of P associate that element which replaces a_1 in that substitution. The group of isomorphisms of P may then be represented as a transitive group in the $p^m - 1$ elements

$$a_2, a_3, \ldots, a_p;$$

when so represented, let it be denoted by R. The transitive group (h) that is generated by the two groups P and R is simply isomorphic to the holomorph of P.

Now to any subgroup of R whose degree is less than $p^m - 1$, there corresponds an imprimitive subgroup of h. For such a subgroup of R would transform some of the substitutions of P into themselves, and these would form an invariant intransitive subgroup of the corresponding subgroup of h.

Further, to any transitive subgroup (R_1) of R whose degree equals $p^m - 1$, there corresponds a primitive subgroup (h_1) of h. For R_1 is the subgroup of h_1 that leaves one of its elements fixed; since this is transitive, it follows that h_1 is primitive

It remains to consider those intransitive subgroups (I) of R that are of degree $p^m - 1$. We note in the first place that if the subgroup (h') of h that corresponds to I is primitive, then any substitution of I besides identity must contain elements from each one of its transitive constituents. Suppose that

some of the substitutions of I that differ from identity do not contain any elements from a given one of its transitive constituents. These substitutions form an invariant subgroup (I') of I whose degree is less than $p^m - 1$. The substitutions of P that correspond to the elements of I that are not found in I', form with identity a subgroup of P. This subgroup is invariant in h', and hence the latter would be imprimitive. Hence, when h' is primitive, I must be formed by establishing a simple isomorphism among its transitive constituents. Every subgroup I of this kind that is contained in R does not give rise, however, to a primitive subgroup of the holomorph. For example, when p equals 2 and m equals 4, R contains a subgroup of order 3 and of degree 15 that gives rise to an imprimitive subgroup of order 48 in the holomorph. The substitutions of P that correspond to the elements in each transitive constituent of the given subgroup of order 3 generate subgroups of order 4. In general, it is evident that the necessary and sufficient condition that h' is primitive is that the elements of each transitive constituent of I contain a set of independent generators of P. If the substitutions that correspond to the elements of any transitive constituent of I generate a subgroup of P of order p^a where α is less than m, then this subgroup is invariant in h' and the latter is imprimitive.

Hence we have this

THEOREM.—*A subgroup* (h') *of the holomorph* (h) *of* P, *that corresponds to a subgroup* (R_1) *of the group of isomorphisms* (R), *is primitive*

(a). *When* R_1 *is a transitive subgroup of degree* $p^m - 1$, or

(b). *When* R_1 *is an intransitive subgroup of degree* $p^m - 1$ *that is formed by establishing a simple isomorphism among its transitive constituents and that is such that the elements of each of its transitive constituents contain a set of independent generators of* P.

Any other subgroup of R *gives rise to an imprimitive subgroup of* h.

Corollary: *When the number of elements in each transitive constituent of* R *is less than* m, *the subgroup* h' *is imprimitive.*

This follows at once from the above theorem, since the number of operators in a set of independent generators of P is m.

6. The theorem just stated is useful in the determination of the primitive groups of degree 16. For it is known that every primitive group of this degree

that does not include the alternating group, contains an invariant abelian subgroup of order 16 and of type (1, 1, 1, 1).*

7. By means of the two preceding theorems we can investigate the number of distinct imprimitive groups G of a given degree that have more than two sets of systems of imprimitivity (the heads differing from identity), and that have all their sets permuted according to primitive groups by the substitutions of G. Any such group is of degree p^{2m}, and the transitive constituents of any of its heads are abelian groups of order p^m and of type $(1, 1, \ldots, 1)$. Also each head is formed by establishing a one-to-one isomorphism among its transitive constituents, the number of these being p^m. Denote the transitive constituents of the head H_1 by the symbols

$$A_1, A_2, \ldots A_{p^m}.$$

It follows that P_1 is a transitive group in these symbols that contains a regular abelian group (P') of type $(1, 1, \ldots, 1)$ as an invariant subgroup. That is, P_1 must be some subgroup of the holomorph (h) of P' that contains P', h being represented as a transitive group in the symbols $A_1, A_2, \ldots, A_{p^m}$. We consider then those imprimitive groups that contain the head H_1 and whose systems of imprimitivity that are determined by H_1 are permuted according to the primitive subgroups of h that contain P'; the groups G must be found among these.

Let the substitutions of H_1 be denoted by the symbols

$$1, S_2, S_3, \ldots, S\ .$$

It is assumed that S_j replaces a_1 by a_j, and that the substitution of the constituent A_i that occurs in S_j is found by replacing the element a_k of the substitution of A_1 that occurs in it by the corresponding element of A_i, where $k = 1, 2, \ldots, p^m$ and i, j are any two of these numbers. The substitutions that interchange the transitive constituents of H_1 in the simplest way according to the substitutions of P, form a second head H_2. Let the substitutions of this head be denoted by the symbols

$$1, t_2, t_3, \ldots, t_{p^m},$$

* Miller, American Journal of Mathematics, Vol. 20 (1898), p. 229.

t_j being that substitution of H_2 that corresponds to the substitution of P' that replaces A_1 by A_j. Now, any third head H_3 must be formed by establishing some simple isomorphism between the substitutions of H_1 and H_2. Without loss of generality, we may assume that the substitutions of H_3 are represented by the symbols

$$1,\ S_2t_2,\ S_3t_3,\ \ldots\ldots,\ S_{p^m}t_{p^m}.$$

For let H_3' denote the group formed by establishing some other isomorphism between H_1 and H_2. A certain permutation of the symbols that denote the substitutions of H_1 will change H_3' into H_3, and corresponding to this permutation there is a definite substitution (S') in the group of isomorphisms of A_1. Let S_1' denote one of the substitutions of the holomorph of A_1 that corresponds to S' (the holomorph being represented transitively of degree p^m), and let S_i' denote the same substitution in the elements of the constituent A_i where $i=2, 3, \ldots, p^m$. Then the substitution $S_1' S_2' \ldots\ldots S_{p^m}'$ transforms H_1 and H_2 into themselves and H_3' into H_3. That is, a group that contains the heads H_1, H_2 and H_3' can be transformed into one that contains the heads H_1, H_2 and H_3.

The largest group within which H_1 is invariant without having its systems of intransitivity interchanged is the group generated by the holomorphs of each of its transitive constituents $A_1, A_2, \ldots\ldots, A_{p^m}$, these being represented as transitive groups of degree p^m. The order of the resulting group divided by p^m gives then the number of sets of substitutions of p^m each that permute according to each substitution of P_1.

When P_1 is regular, that is when G is regular, it is clear that m must equal unity. If m were greater than unity, G would contain systems of imprimitivity that are not permuted by its substitutions according to primitive groups.

When P_1 is not regular, let S' denote any substitution in P that is not contained in P'; and let t' denote the substitution that interchanges in the simplest manner the transitive constituents of H_1 according to S'. The substitution S' transforms the substitutions of P' according to a certain operator in the group of isomorphisms of P'; and t' transforms the substitutions of H_2 in exactly the same way. Further, t' is commutative with each substitution of H_1. Suppose now that any one of the substitutions which with H_1 generate the sets of substitutions that permute according to S' be denoted by

$$s_1 s_2,\ \ldots,\ s_{p^m} t', \qquad \text{(A)}$$

where s_i is some substitution in the holomorph of A_i, i being any of the numbers 1, 2, ..., p^m. Since the substitution (A) must transform H_2 into itself, it follows that

$$s_1 s_2, \ldots, s_{p^m} \tag{B}$$

must be commutative with each of the substitutions t_i, where i is as above. Hence s_i must be the same substitution in the elements of A_i as s_j is in the elements of A_j, i and j being any two of the numbers 1, 2, ..., p^m. The number of substitutions (B) is then not greater than the number of operators in the group of isomorphisms of A_1. Further, since (A) must transform H_3 into itself, it follows that the substitution (B) must transform the substitutions of H_1 in exactly the same way as t' transforms the substitutions of H_2. That is, (B) is a fixed substitution. It is clear also that the corresponding substitution (A) thus found has its proper power in the head H_1. Hence there is one and only one generating substitution that permutes according to S' that transforms H_1, H_2 and H_3 into themselves respectively, and that has its proper power in H_1. That is, there is just one imprimitive group of the given kind that is isomorphic to any primitive group P_1. The number of primitive groups that contain P' is given by the preceding theorem, so that we have determined now the number of imprimitive groups G of degree p^{2m}. Of the groups G thus found, it is evident that to conjugate subgroups (R_1) of the group of isomorphisms correspond conjugate groups G. We have then the

THEOREM.—*The number of imprimitive groups G of a given degree that contain more than two sets of systems of imprimitivity (the heads differing from identity) and for which P_1, P_2, ... are primitive groups, is as follows:*

(a). *When G is regular, there is just one such group; its degree is p^2.*

(b). *When G is not regular, the number is equal to the number of distinct primitive groups that are contained in the holomorph (h) of the abelian group P' and that contain P'.*

It is well known that the group of isomorphisms of an abelian group of order p^m and of type (1, 1, ..., 1) is simply isomorphic to the linear homogeneous group.* It appears then that the study of the imprimitive group G of the above theorem is closely associated with that of the linear homogeneous group.

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* Moore, Bulletin of the American Mathematical Society, Vol. 2 (1895), p. 34.

8. We proceed now to investigate the number of sets of systems of imprimitivity that belong to the groups just determined. When G is regular, it is the non-cyclic group of order p^2 and so contains $p+1$ sets of systems—this being the number of subgroups differing from identity that G contains.

Suppose then that G is a non-regular group. Let R_1 denote the subgroup of P_1 that leaves any element fixed. We note first that G does not admit the head identity. The subgroup (G_1) of G that leaves any element fixed has one and just one substitution in common with any division of G—the divisions being formed with respect to the subgroup that contains the heads $H_1, H_2, \ldots$. If now G admits the head identity, then G_1 must be contained in a larger subgroup of G which contains no invariant subgroup of G. This is evidently not the case. Any head of G that differs from H_1 and H_2 is then formed by establishing some simple isomorphism between these two heads. Denote by H_4 any such isomorphisms that differ from H_3. Our problem is to determine under what conditions H_4 will be transformed into itself by the substitutions of G. Corresponding to H_4 there is a definite isomorphism of H_2 to itself, viz., the one formed by replacing each S in H_4 by the corresponding t. And this again corresponds to a definite isomorphism of P' to itself. That is, to H_4 there is associated in this way a definite substitution (r) of the group of isomorphisms of P'. Also when H_4 is transformed into itself by the substitution of G, it is clear that r must be transformed into itself by the substitutions of R_1; and conversely. It follows, therefore, that the number of heads H_4 is equal to the number of substitutions differing from identity in the group of isomorphisms of P' that are commutative to each substitution of R_1. To identity corresponds the head H_3. Hence the

THEOREM.—*The number of sets of systems of imprimitivity of the groups G in the preceding theorem is as follows:*

(a). *When G is regular, there are $p+1$ different sets.*

(b). *When G is non-regular, there are $c+2$ different sets, where c is the number of substitutions in the group of isomorphisms of P' that are commutative with each substitution of R_1.*

9. We consider now the imprimitive groups G that contain more than one set of systems of imprimitivity, each set being permuted by the substitutions of

G according to a primitive group, and that have all their heads identity except one.

Suppose in the first place that G denotes a regular group of this kind. We know that every regular group contains a set of systems of imprimitivity that is permuted by its substitutions according to any transitive representation of the group. It follows then that G must be a group that can be represented in the imprimitive form only when its degree equals its order, and that can be represented in the primitive form of a lower degree. The only groups that satisfy these requirements are the non-cyclic groups of order pq, where p and q are distinct primes.* If $p > q$, these groups contain $q + 1$ subgroups, not including identity and hence G contains $q + 1$ sets of systems of imprimitivity. The heads that correspond to q of these are identity and the remaining head is of order p. Hence we have the

THEOREM.—*The only regular groups G that contain more than one set of systems of imprimitivity, each set being permuted by the substitutions of G according to a primitive group, and that have only one head different from identity, are the non-cyclic groups of order pq, where p and q are distinct primes. If $p > q$, these groups have $q + 1$ sets of systems of imprimitivity.*

10. Suppose next that the subgroup (G_1) of G that leaves one element fixed is contained in the head (H) that differs from identity. When G_1 is of order unity, the groups G are determined by the preceding theorem. Also when G is regular, the subgroup G_1 is maximal in H. We prove now that this is true when G is non-regular.

If G_1 is not maximal in H, there is a subgroup H_1 of H in which G_1 is maximal. This subgroup H_1 determines a set of systems of imprimitivity of G. Two cases arise according as H_1 contains an invariant subgroup of G or does not contain such a subgroup. In the former case the set of systems that corresponds to H_1 is left unchanged by the substitutions of the invariant subgroup contained in H_1. The group G would contain then two heads that differ from identity; this is contrary to the assumption made. In the latter case the set of systems that corresponds to H_1 is permuted by the substitutions of G according to a group (P)

* Dyck, Mathematische Annalen, Vol. 22 (1883), p. 101.

that is simply isomorphic to G. Since H_1 is not maximal in G, the group P would be imprimitive. It follows, therefore, that G_1 is a maximal subgroup of H.

Suppose now that H is contained in a subgroup (H') of G whose order exceeds that of H. The subgroup H' gives rise to a set of systems of imprimitivity that are found by uniting the systems that correspond to H into larger systems. This new set of systems is left unchanged by the substitutions of H, and so in this case G contains two heads that differ from identity. It follows accordingly that H is a maximal subgroup of G. That is, the quotient group G/H is of prime order p.

The systems of imprimitivity that are left unchanged by H are permuted by the substitutions of G according to a primitive group that is simply isomorphic to the quotient group G/H. Since these systems are the systems of intransitivity of H, it follows that the number of transitive constituents of H is equal to p.

Consider now a set of systems that is not left unchanged by any substitution of G besides identity, and let P denote the primitive group according to which this set of systems is permuted by the substitutions of G. The subgroup P' of P that corresponds to the subgroup H of G is transitive. Suppose now that H is not formed by establishing a simple isomorphism among its transitive constituents. It contains then an invariant subgroup whose degree is less than the degree of H. The subgroup of P' that corresponds to this invariant subgroup of H would be of a lower degree than the degree of P; also it would be invariant in P'. This, however, cannot be true, since an invariant subgroup of a transitive group is of the same degree as that of the group. Hence H is formed by establishing a simple isomorphism among its transitive constituents.

The subgroup of G that gives rise to the primitive representation P just considered, must contain G_1 as an invariant subgroup, and its order is equal to pg_1, where g_1 is the order of G_1. Its order could not be greater than pg_1 since then G_1 would not be maximal in H. It follows that the degree of P is equal to the degree of each of the transitive constituents of H, so that to each element of P there are associated p elements of G. Also since G_1 is maximal in H, it follows that P' is a primitive group.

When G is not regular, there is just one subgroup besides H that contains G_1. That is, there is just one set of systems that is permuted by the substitu-

tions of G according to a simply isomorphic group. It follows, therefore, that when G is not regular, it contains just two sets of systems of imprimitivity.

These results may be summed up in the

THEOREM.—*If G denotes an imprimitive group having more than one set of systems of imprimitivity, each set being permuted according to a primitive group by the substitutions of G, and if just one of the heads (H) differs from identity, then when the subgroup (G_1) of G that leaves any element fixed, is contained in H.*

(a). *The head H is maximal in G, the quotient group G/H being of prime order p. Also H is formed by establishing a simple isomorphism among its transitive constituents—the number of these being p.*

(b). *The subgroup G_1 is maximal in H.*

(c). *G has just two sets of systems when it is not regular. The one whose head is identity contains p elements in each system.*

11. It is not difficult to construct general classes of groups that come under those defined in the preceding theorem.

Suppose, for example, that the transitive constituents of H are alternating groups of degree $n\,(n > 2)$ and that H is formed by writing after each substitution of the first constituent the same substitution in the other constituent. H has just two transitive constituents. A substitution which with H generates a group G of the kind in question cannot be the ordinary t; for this would be a commutative substitution and the resulting group could not be represented in primitive form. Let s_1 denote any negative substitution of the symmetric group of degree n that contains the first transitive constituent of H; and let s_2 denote the same substitution in the elements of the second constituent. Then

$$s_1 s_2 t \tag{A}$$

is a substitution which with H generates a group G of the desired kind. Further, there is just one group of this kind. For any other substitution that could be used in place of (A) would be of the form

$$s_1 S_2 s_2 t\,,$$

where S_2 is some substitution in the elements of the second constituent that is commutative with each substitution of this constituent. Since the constituents are primitive groups there is no such substitution except when $n = 3$. In this latter case the group is regular and there is just one.

A second class of groups G is obtained in a similar way if we take for the transitive constituents of H semi-metacyclic groups of degree q. The groups thus obtained are of degree $2q$, and they are simply isomorphic to the metacyclic groups of degree q.

12. Let, finally, G denote an imprimitive group that contains no head that differs from identity and that has all its sets of systems of imprimitivity permuted by its substitutions according to primitive groups. Denote by G_1 the subgroup that leaves a given element unchanged.

The simple group of order 60, when represented as a transitive group of degree 12, illustrates the occurrence of a group of this kind that has just one set of systems. When the same group is represented as a transitive group of degree 20, it has two sets of systems, both of which are permuted according to primitive groups. In these examples the subgroup G_1 is maximal in larger subgroups of G. In general, it is evident that *the subgroup* G_1 *must be maximal in any larger subgroup* (G_2) *of* G *in which it is found, and, further, that the subgroup* G_2 *cannot contain any invariant subgroup of* G *besides identity.* It is clear that G cannot be regular. Also its order cannot be the power of a prime. The number of sets of systems that belong to any such group is equal to the number of subgroups G_2 that it contains.

13. THEOREM.—*Every imprimitive group* G *that admits only the heads identity is insolvable.*

Since G admits only identity as a head, it must be possible to represent it as a primitive group (P). If now G is solvable, so also is P. Suppose that G is solvable. Then P must be of degree p^a, where p is a prime, and it contains as a minimal invariant subgroup an abelian group (P_1) of order p^a and type $(1, 1, \ldots, 1)$.* To the subgroup P_1 of P there corresponds an invariant subgroup of G; and since P_1 is regular and of degree p^a, it follows that this subgroup is also intransitive. Hence G in this case contains a head that differs from identity, and this contradicts our hypothesis.

* Galois, Oeuvres Mathématiques, p. 27. Cf. also Jordan, Traité des Substitutions, p. 398.

SECTION II.—*The substitutions which are commutative with each substitution of any transitive group.*

1. THEOREM.—*Let G denote any transitive group of degree n and order g. The number of substitutions in the elements of G that are commutative with each of its substitutions is equal to the order of the quotient group H/G_1, where G_1 is a subgroup of G that leaves any element fixed, and H is the largest subgroup of G that contains G_1 self-conjugately.*

It is well known that this theorem is true when G is regular.* To prove that it is true for any transitive group G, let the substitutions of the group be denoted by

$$1,\ S_2,\ S_3,\ \ldots\ldots,\ S_g,$$

and the elements of these substitutions by

$$a_1,\ a_2,\ \ldots\ldots,\ a_n.$$

Suppose that G_1 is that subgroup of G which leaves a_1 fixed, and let its substitutions be denoted by

$$1,\ S_2,\ S_3,\ \ldots\ldots,\ S_{g_1},$$

where g_1 is the order of G_1.

The substitutions of G may be arranged in the following rectangular array:

$$\left.\begin{array}{lllll} 1, & S_2, & \ldots\ldots, & S_{g_1}, & a_1, \\ S_{g_1+1}, & S_2S_{g_1+1}, & \ldots\ldots, & S_{g_1}S_{g_1+1}, & a_2, \\ & & \ldots\ldots & & \\ S_{g_1+n-1}, & S_2S_{g_1+n-1}, & \ldots\ldots, & S_{g_1}S_{g_1+n-1}, & a_n. \end{array}\right\} \qquad \mathrm{I}$$

In this array the substitutions of the i^{th} row replace a_1 by a_i where i is any of the numbers $1, 2, \ldots\ldots, n$. Also if the i^{th} row is denoted by a_i, then any substitution S_x of G is given by the permutation on the symbols associated with the rows which arises when all the substitutions of the array are multiplied by S_x.† If the substitution S_x is commutative with each substitution of G, then evidently the same permutation of the rows will take place whether pre-multiplication or post-multiplication is made use of.

* Jordan, Journal de l'École Polytechnique, Vol. 22 (1861), p. 153; cf. Traité des Substitutions, p. 60.
† Miller, Bulletin of the American Mathematical Society, 2d series, Vol. 3 (1896), p. 214.

We prove now that any substitution C, which is not found in G and which is commutative to each of the substitutions of G, is given by the permutation on the symbols associated with the rows that arises when some substitution of G is multiplied by all its substitutions, i. e., by pre-multiplication of some substitution of G. Since C is not found in G, it will generate with G a larger transitive group G' in the same elements. The subgroup G_1' of G' that leaves a_1 fixed will contain G_1. Let S_tC denote a substitution which with G_1 will generate G_1', S_t being some substitution of G. As above, the substitutions of G' may be arranged in the following array:

$$\left.\begin{array}{llllll} 1, & S_2, & \ldots, S_{g_1}, & S_tC, & \ldots, & a_1, \\ S_{g_1+1}, & S_2S_{g_1+1}, & \ldots, S_{g_1}S_{g_1+1}, & S_tCS_{g_1+1}, & \ldots, & a_2, \\ \vdots & & \ldots & & \ldots & \\ S_{g_1+n-1}, & S_2S_{g_1+n-1}, & \ldots, S_{g_1}S_{g_1+n-1}, & S_tCS_{g_1+n-1}, & \ldots, & a_n. \end{array}\right\} \quad \text{II}$$

The substitution C is contained in this array, and, as noted above, if the pre-multiplication of C upon the substitutions of G is performed, the resulting permutation on the elements denoting the rows is identical with C. If pre-multiplication be used with reference to any substitution in the first row of the array, then each row goes into itself. This follows at once from the fact that tthe substitutions in the i^{th} row replace a_1 by a_i, where i equals $1, 2, \ldots, n$.

Suppose now that the substitution $S_t^{-1}C^{-1}C$ is multiplied by each substitution of G'. From what has just been said, it follows that the resulting permutation on the letters associated with the rows is identical with C. The substitution $S_t^{-1}C^{-1}C$ is identical with S_t^{-1}, and hence we have proved our statement.

We prove now that the number of substitutions of G which by pre-multiplication give rise to distinct permutations on the elements associated with the rows of the array I, is equal to the order of the quotient group H/G_1, where H is the largest group that contains G_1 self-conjugately. Let S_iG_1 and G_1S_i denote the result of pre-multiplication and post-multiplication of G_1 with S_i respectively. Then if S_t by pre-multiplication gives rise to a permutation C of the rows of I, we must have

$$S_tG_1 \equiv G_1S_{g_1+k},$$

$$\therefore \qquad S_tG_1S_t^{-1} \equiv G_1S_{g_1+k}S_t^{-1} \equiv G_1S_r, \qquad (S_r = S_{g_1+k}S_t^{-1}).$$

The right-hand member of this identity consists of the substitutions found in some row. And since the left-hand member contains the identical substitution, it follows that S_t must transform G_1 into itself. Conversely, if S_t transforms G_1 into itself, it gives rise to a permutation of the rows of I.

It has been seen above that, if S_t is a substitution of G_1, the corresponding substitution C is the identical substitution. When S_t is any substitution of H that is not found in G_1, the corresponding substitution C will be different from identity. Also all the substitutions of H that are found in the same row with S_t will give rise to the same substitutions C, since these substitutions are found by multiplying those of G_1 by S_t.

Finally, the substitutions C that correspond to substitutions S_t that are found in different rows of H are distinct. For each replaces a_1 by the element associated with the row in which it is found. The number of substitutions C is then equal to h, the order of the quotient group H/G_1. It is clear also that the substitutions found in this way are all commutative with each substitution of G.

Since H is a subgroup of G, it follows that the number of rows contained in H is a divisor of the number contained in G. Hence

Corollary I.—*The number of substitutions that are commutative with each substitution of G is equal to some divisor of the degree of G.*

When G is a primitive group, the subgroup G_1 that leaves one element fixed, is maximal, and in this case the order (h) of H/G_1 is equal either to unity or to g. Hence we have

Corollary II.—*Identity is the only substitution that is commutative with each substitution of a primitive group of composite order.*

2. If α denotes the number of letters left fixed by G_1, then n/α is the number of subgroups in the conjugate set to which G_1 belongs. The number of substitutions, x, that transform G_1 into itself is the same as the number that transforms it into any one of its conjugates.

$$\therefore \qquad x\,.\,n/\alpha = g = ng_1,$$

or

$$x = \alpha g_1,$$

or

$$x/g_1 = h = \alpha.$$

The above theorem may therefore be stated in this form:

The number of substitutions in the elements of any transitive group G that are commutative with each of its substitutions is equal to the number of elements that are left unchanged by the subgroup G_1 that leaves any element of G unchanged.

When the order of G is the power of a prime, then the order of the quotient group H/G_1 is also the power of the same prime. Hence we have

Corollary I.*—*If the order of G is the power of a prime, then the number of elements left unchanged by G_1 is a power > 1 of the same prime.*

3. When G is a regular group, the substitutions that are commutative with each of its substitutions, form a group that is simply isomorphic to G. So in this case the order of each substitution that is commutative to all the substitutions of G is equal to the order of some substitution of G. When G is not regular, this is still true. For, from the method of forming any substitution C that is commutative to all the substitutions of G, it follows that to C there corresponds a definite substitution in the associate of G whose order equals that of C. We have then the

THEOREM.—*The order of any substitution that is commutative with each substitution of a transitive group is equal to the order of some substitution of that group.*

4. Let the associate of a group G whose order is g be denoted by G'. If G contains no invariant substitution, then it contains no substitutions in common with G. In this case G and G' generate a group $\{G, G'\}$ whose order equals g^2 and whose degree equals g. *The subgroup G_1 that leaves any element fixed in $\{G, G'\}$, is formed by establishing some simple isomorphism between G and G'.*

For, suppose the substitutions of $\{G, G'\}$ are written in rectangular array in such a way that the substitutions of G form the first row. The subgroup G_1 cannot have more than one substitution in common with any row of this array. If it had two, then one times the inverse of the other would belong to G, and this could not be true since this product would be a substitution differing from identity that is found in G. The group G_1 contains, therefore, just one substitution from each row of our array. The substitutions in any row are found by multiplying the g substitutions of G by some substitution of G'. Further, no

* Miller, American Journal of Mathematics, Vol. 23 (1900), p. 173.

two substitutions of G_1 can involve the same substitution of G; for then one times the inverse of the other would be of degree g. It follows directly that G_1 consists of a simple isomorphism between G and G'.

5. By means of the statement just proved in reference to G_1, we can give an easy proof by substitution theory of the fundamental theorem that every group whose order is the power of a prime, contains invariant substitutions. Suppose that such a group G of order p^a does not contain any invariant substitution. Then $\{G, G'\}$ is of order p^{2a}, and G_1 is formed by establishing some simple isomorphism between G and G'. The subgroup G_1 in this case leaves a multiple of p elements fixed and so is transformed into itself by some substitution S of G. This substitution S is commutative with each substitution of G', and hence must also be commutative with each substitution of G. It follows that our hypothesis is wrong, and so G contains invariant substitutions.

SECTION III.—*Some theorems relating to the construction of imprimitive groups.*

1. In constructing the imprimitive groups of a given degree by means of tentative processes, the number of trials that needs to be made is frequently large, and much of the work is merely repetition. Further, it frequently happens that the method by means of which certain groups of a given degree can be found, needs little change to determine corresponding groups of other degrees. It seems desirable then to establish general theorems which apply to groups of different degrees. Several theorems of this nature have already been proved.* Of these, one of the most useful states that "there is only one imprimitive group whose head is the product of the groups obtained by writing a given group in the different systems of elements, and which permutes the systems according to a given cyclical substitution."

The theorems in paragraph 2 to 7 will be found useful in constructing the imprimitive groups of degree np, where n is any integer greater than 2, and p any prime number. Let G' denote the symmetric group in the elements $a_1, a_2, \ldots, a_n$; G'', the symmetric group in the elements $b_1, b_2, \ldots, b_n$; $\ldots$; G^p, the sym-

* Miller, Quarterly Journal of Mathematics, Vol. 28 (1895), p. 193; American Journal of Mathematics, Vol. 21 (1899), p. 295.

metric group in the elements $m_1, m_2, \ldots\ldots, m_n$. Also let P denote the metacyclic group of degree p in the letters $A, B, C, \ldots\ldots, M$ and P_{i_2} that invariant subgroup of P whose index under P is i_2. When i_2 is different from $p-1$, the group P_{i_2} may be generated by a substitution (p_1) of order p and a substitution (p_2) of order $\frac{p-1}{i_2}$; when i_2 equals $p-1$, the group P_{i_2} is generated by a single substitution of order p. The generator p_1 may be taken as the substitution $ABC \ldots\ldots M$, and it may be assumed that the generator p_2 does not contain the letter A. When p equals 2, P will denote the group (AB). The substitutions that permute the systems in the simplest way according to p_1 and p_2 will be denoted by t_1 and t_2 respectively. That is,

$$t_1 \equiv a_1b_1c_1 \ldots\ldots m_1 \,.\, a_2b_2c_2 \ldots\ldots m_2 \,.\, \ldots\ldots \,.\, a_nb_nc_n \ldots\ldots m_n,$$

and t_2 is a substitution of order $\frac{p-1}{i_2}$ that does not involve the elements $a_1, a_2, \ldots\ldots, a_n$.

2. THEOREM.—*The number of imprimitive groups of degree np that contain the head*

$$H \equiv \{G', G'', \ldots\ldots, G^p\} \text{ pos*}$$

and whose substitutions permute the systems of intransitivity of H according to P_{i_2} *is as follows:*

(a). *When* $p = 2$, *there are two such groups.*

(b). *When* $p > 2$, *there is one group if* $\frac{p-1}{i_2}$ *is odd and two groups if* $\frac{p-1}{i_2}$ *is even.*

The largest group within which the given head is invariant without having its systems of intransitivity interchanged is $\{G', G'', \ldots\ldots, G^p\}$. Hence there are just two sets of substitutions that transform according to any substitution of P_{i_2}. Those that permute according to p_1 may be obtained by multiplying the substitutions in the head by

$$t_1 \text{ and } a_1a_2 \,.\, t_1.$$

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* The notation used is that given by Cayley, Quarterly Journal of Mathematics, Vol. 25 (1890–1), p. 71.

Both of these transform H into itself. The p^{th} power of t_1 is identity, while that of $a_1a_2 \,.\, t_1$ is $a_1a_2 \,.\, b_1b_2 \ldots\ldots m_1m_2$. This latter is found in the head only when $p = 2$. Hence there is just one imprimitive group that contains the given head and that corresponds to P_{p-1} when $p > 2$. When $p = 2$, there are two such groups; these are distinct, since one contains negative substitutions while the other does not.

When i_2 is less than $p - 1$, the substitutions that permute according to p_2 may be obtained by multiplying the substitutions of H by

$$t_2 \text{ and } a_1a_2 \,.\, t_2 .$$

Since there is just one group that corresponds to P_{p-1}, each group that corresponds to P_{i_2} must contain this, and hence we may assume that t_1 is found in each such group. The substitutions t_2 and $a_1a_2 \,.\, t_2$ both transform the head and also the group that corresponds to P_{p-1} into themselves. The $\frac{p-1}{i_2}$th power of t_2 is identity, and hence there is always one group that corresponds to P_{i_2}. The $\frac{p-1}{i_2}$th power of $a_1a_2 \,.\, t_2$ is identity or a_1a_2 according as $\frac{p-1}{i_2}$ is even or odd. Hence $a_1a_2 \,.\, t_2$ may be used only when $\frac{p-1}{i_2}$ is even. The corresponding group is distinct from the one obtained when t_2 is taken, as the one contains negative substitutions while the other does not. Hence, when $\frac{p-1}{i_2}$ is even there are two groups that correspond to P_{i_2}, and when $\frac{p-1}{i_2}$ is odd there is just one such group.

3. THEOREM.—*When $p > 2$, there is just one imprimitive group of degree np that contains the head*

$$H \equiv G' \text{ pos } G'' \text{ pos } \ldots\ldots G^p \text{ pos} + G' \text{ neg } G'' \text{ neg } \ldots\ldots G^p \text{ neg}$$

and whose substitutions interchange the systems of intransitivity of H according to P_{i_2}. When $p = 2$, there are two such groups.

The largest group within which H is invariant without having its systems interchanged is $\{G', G'', \ldots\ldots, G^p\}$, and hence there are 2^{p-1} sets of substitutions that transform according to a given substitution of P_{i_2}, The 2^{p-1} sets

that transform according to p_1, may be found from H and the substitutions obtained by multiplying each substitution in the group

$$(s_1)(s_2) \ldots . (s_{p-1}) \tag{A}$$

by t_1, where $s_1 = a_1a_2$, $s_2 = b_1b_2$, etc. Each of these transforms H into itself and each has its p^{th} power in H. The resulting groups, however, are all conjugate when p is greater than 2. For

$$s_3s_5 \ldots . s_pt_1s_3s_5 \ldots . s_p = s_2s_3 \ldots . s_pt_1.$$

That is, the group $\{H, t_1\}$ can be transformed into the group $\{H, s_1t_1\}$. Similarly, it can be transformed into the groups $\{H, s_it_1\}$, where $i = 2, 3, \ldots, p-1$ and, therefore, into all the others. Hence when $p > 2$, there is just one group that corresponds to P_{p-1}. It may be taken as $\{H, t_1\}$, and we may assume that it is found in each group that corresponds to $P_{i_2} (i_2 < p - 1)$. When $p = 2$, the head contains only positive substitutions. In this case the groups that correspond to P are distinct, since the one contains negative substitutions while the other does not.

By multiplying each substitution in the group (A) by t_2, we obtain a set of substitutions which with H generate the sets that transform according to p_2. Each of these transforms the head into itself, but t_2 is the only one that transforms $\{H, t_1\}$ into itself. For if we transform t_1 by the inverse of any other one, $s_is_j \ldots . t_2$, say, we have

$$s_is_j \ldots . t_2t_1t_2^{-1}s_is_j \ldots . = s_is_j \ldots . s_{i'}s_{j'} \ldots . t_1^a,$$

where $t_2t_1t_2^{-1} = t_1^a$, and i', j' are certain ones of the subscripts $1, 2, \ldots, p$. There is an even number of s's in the part of this substitution that precedes t_1^a; further this part cannot be identity. Hence this substitution is not found in $\{H, t_1\}$, and, therefore, there is just one group that corresponds to P_{t_2} when $p > 2$.

4. THEOREM.— *When* $\frac{p-1}{i_2}$ *is even, there are two imprimitive groups of degree np that contain the head*

$$H \equiv G' \text{ pos } G'' \text{ pos } \ldots . G^p \text{ pos}$$

and whose substitutions interchange the systems of intransitivity of H *according to* P_{t_2}. *Where* $\frac{p-1}{i_2}$ *is odd, there is just one such group.*

The largest group within which H is invariant without having its systems interchanged is $\{G', G'', \ldots, G^p\}$. There are then 2^p sets of substitutions that transform according to a given substitution of P_{i_2}. We know that there is just one distinct group that contains this head and corresponds to a cyclic substitution of order p. Hence it may be assumed that each group that contains H and that corresponds to P_{i_1} contains the substitution t_1.

The substitutions that permute according to p_2 may be found by multiplying H by the substitutions obtained by affixing t_2 to each substitution in the group

$$(s_1)(s_2) \ldots (s_p),$$

where s_i has the same meaning as in the preceding theorem. Of these generators t_2 and $s_1 s_2 \ldots s_p t_2$ are the only ones that transform $\{H, t_1\}$ into itself. The former has its $\frac{p-1}{i_2}$th power in H and generates with $\{H, t_1\}$ a group that corresponds to P_{i_2}. The latter has its $\frac{p-1}{i_2}$th power in the head only when $\frac{p-1}{i_2}$ is an even number. It transforms both H and $\{H, t_1\}$ into themselves. Hence, when $\frac{p-1}{i_2}$ is even, there are two groups that contain the given head and that correspond to P_{i_2}. The corresponding groups are distinct, since one contains negative substitutions while the other does not.

5. THEOREM.—*The number of imprimitive groups of degree np that contain the head*

$$H \equiv \{G' \text{ pos}, G'' \text{ pos}, \ldots, G^p \text{ pos}\}_{1, 1, \ldots, 1}$$

and whose substitutions interchange the systems of intransitivity of H according to P_{i_2} is as follows:

(a). *When $p = 2$, there are two groups.*

(b). *When $p = 3$ and $n = 3$, there are two groups if $i_2 = p - 1$, and three groups if $i_2 < p - 1$.*

(c). *When $p > 3$ and $n = 3$ or $p > 2$ and $n > 3$, there are two groups if $\frac{p-1}{i_2}$ is even and there is one group if $\frac{p-1}{i_2}$ is odd.*

In this case H is formed by writing after each substitution of G' pos the same substitutions in the other sets of elements.

(a). When $p = 2$ and $n = 3$, there are evidently two distinct groups. When $p = 2$ and $n > 3$, the only substitutions that permute according to p_1 and that also transform H into itself, may be obtained by multiplying H by the substitutions t_1 and $s_1 s_2 t_1$. Both of these have their squares in the head and they generate with H groups that are clearly distinct.

(b). When $p = 3$ and $n = 3$, the groups that are isomorphic to P_{p-1} are of order and degree 9. It follows that there are two such groups, since there are two distinct abstract groups of order p^2. They may be written $\{H, t_1\}$ and $\{H, a_1 a_2 a_3 t_1\}$.

The substitutions that permute according to p_2 and that transform both H and $\{H, t_1\}$ into themselves may be obtained from the head by means of the substitutions

$$\begin{array}{rl} t_2, & s_1 s_2 s_3 t_2, \\ S_1 S_2^2 t_2, & S_1 S_2^2 s_1 s_2 s_3 t_2, \\ S_1^2 S_2 t_2, & S_1^2 S_2 s_1 s_2 s_3 t_2, \end{array}$$

where $S_1 = a_1 a_2 a_3$, $S_2 = b_1 b_2 b_3$, etc. Each of these has its square in the head except the last two in the second column. Those in the first column clearly generate with $\{H, t_1\}$ conjugate groups. The groups $\{H, t_1, t_2\}$ and $\{H, t_1, s_1 s_2 s_3 t_2\}$ are distinct since one contains only positive substitutions while the other contains negative substitutions.

The group $\{H, a_1 a_2 a_3 t_1\}$ is a cyclic group of order 9. If a group that corresponds to P contains it self-conjugately, then that group is of order 18. There is just one such non-abelian group of this order,* and it can be represented in only one way as an imprimitive group of degree 9. Hence there is just one group with the head H that contains $\{H, a_1 a_2 a_3 t_1\}$ self-conjugately and that corresponds to P.

(c). When $p > 3$ and $n = 3$, the groups that are isomorphic to P_{p-1} are of order and degree $p \cdot 3$. Since $p > 3$ and the subgroup of order 3 is self-conjugate, it follows that there is just one group that contains H and that corresponds to P_{p-1}. It may be written $\{H, t_1\}$.

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* Cole and Glover, American Journal of Mathematics, Vol. 15 (1893), p. 206.

The general substitution that permutes according to p_2 may be taken of the form

$$S_1^{\alpha_1}S_2^{\alpha_2}\ldots\ldots S_p^{\alpha_p}(s_1s_2\ldots\ldots s_p)^{\beta}\,t_2, \qquad \text{(A)}$$

where $\alpha_1, \alpha_2, \ldots\ldots, \alpha_p = 0, 1, 2$ and $\beta = 0, 1$. If this substitution transforms $\{H, t_1\}$ into itself, then $S_1^{\alpha_1}S_2^{\alpha_2}\ldots\ldots S_p^{\alpha_p}$ must also do so. Now

$$S_1^{\alpha_1}S_2^{\alpha_2}\ldots\ldots S_p^{\alpha_p}t_1,\quad S_1^{-\alpha_1}S_2^{-\alpha_2}\ldots\ldots S_p^{-\alpha_p} = S_1^{\alpha_1-\alpha_2}S_2^{\alpha_2-\alpha_3}\ldots\ldots S_p^{\alpha_p-\alpha_1}t_1.$$

If this be in $\{H, t_1\}$, it must be of the form

$$(S_1S_2\ldots\ldots S_p)^{\alpha}\,t_1.$$

That is, we have the relations

$$\alpha_i - \alpha_1 \equiv -(i-1)\,\alpha \qquad (\text{mod } 3),$$

where $i = 1, 2, \ldots\ldots, p$ and $\alpha = 0, 1, 2$. Putting $i = p$, we see that

$$\alpha p \equiv 0 \qquad (\text{mod } 3).$$

Hence zero is the only permissible value of α and of the substitutions (A), we need consider only

$$t_2 \text{ and } s_1s_2\ldots\ldots s_pt_2,$$

The first of these has its $\frac{p-1}{i_2}$th power in the head and hence generates with $\{H, t_1\}$ a group that corresponds to P_{i_2}. The second has its $\frac{p-1}{i_2}$th power in the head only when $\frac{p-1}{i_2}$ is an even number.

Therefore, when $\frac{p-1}{i_2}$ is even, there are two groups that contain the given head and that correspond to P_{i_2}. They are distinct, since one contains only positive substitutions while the other contains only negative ones.

When $p > 2$ and $n > 3$, the substitutions that permute like p_1 and that transform H into itself, may be found by multiplying H by the substitutions

$$t_1 \text{ and } s_1s_2\ldots\ldots s_pt_1.$$

Of these only the first has its p^{th} power in the head, and hence there is just one group that contains the given head and is isomorphic to P_{p-1}. The remain-

ing part of the proof for this case is the same as the latter part of the preceding one.

6. The group $\{G', G'', \ldots, G^p\}$ may also be used as a head. It contains all the substitutions which transform it into itself without interchanging its systems of intransitivity. Hence it is contained as a head in just one imprimitive group whose substitutions permute its systems of intransitivity according to a given transitive group. The same remark applies to the head $\{G', G'', \ldots, G^p\}_{1, 1, \ldots, 1}$.*

7. The remaining theorems in this section apply to imprimitive groups of degree pq where p and q are prime numbers which may be the same or different primes.

Let G_1 denote the metacyclic group in the elements $a_1, a_2, \ldots, a_q$; G_2, the metacyclic group in the elements $b_1, b_2, \ldots, b_q$; ....; G_p, the same group in the elements $m_1, m_2, \ldots, m_q$; and let G_{i, i_1} denote that invariant subgroup of G_i whose index under G_i is i_1. The symbols P, P_{i_2}, p_1, p_2, t_1 and t_2 will be used as they were in the theorems just given. We shall assume further that the group G_1 is generated by the substitution $S_1 = a_1 a_2 a_3 \ldots a_q$ of order q, and a substitution S_1 of order $q-1$ in the $q-1$ elements $a_2, a_3, a_4, \ldots, a_q$. The symbols S_i and s_i will denote the same substitutions in the elements of the group G_i, where $i = 2, 3, \ldots, p$, as S_1 and s_1 do in the elements of G_1.

8. THEOREM.—*The number of imprimitive groups of degree pq that contain the head*

$$H \equiv \{G_{1, i_1}, G_{2, i_1}, \ldots, G_{p, i_1}\}$$

and whose substitutions interchange the systems of intransitivity of H according to P_{i_2}, is equal to the number of solutions of the congruences

$$\beta\left(\frac{p-1}{i_2}\right) \equiv h i_1 \pmod{q-1},$$

where β is restricted to the values 0, 1, 2, ..., $i-1$ and *h is any integer.*

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* Miller, Quarterly Journal of Mathematics, Vol. 28 (1895), p. 195.

The largest group within which H is invariant without having its systems interchanged is $\{G_1, G_2, \ldots, G_p\}$. There are accordingly i_1^p sets of substitutions that transform according to each substitution of P_{i_2}.

Since the given head is the direct product of p transitive groups written in different sets of elements, we know that there is just one distinct group that contains this head and that corresponds to P_{p-1}. This may be taken as $\{H, t_1\}$ and we may assume that each group to be found contains this as a self-conjugate subgroup.

As $s_i^{i_1}$ is the lowest power of s_i besides s_i^0 that occurs in G_{t, i_1}, it follows that the i_1^p sets of substitutions that permute according to p_2 may be obtained by multiplying the head by the substitutions

$$s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p} t_2, \tag{A}$$

where $a_1, a_2, \ldots, a_p = 0, 1, 2, \ldots$, or $i_1 - 1$. For it is clear that these sets contain no common substitutions, and since each exponent may have i_1 different values, there are i_1^p of them. Each of the substitutions (A) that can be used must transform $\{H, t_1\}$ into itself. This will be true when $s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p}$ transforms t_1 into a substitution in $\{H, t_1\}$. The substitution

$$s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p} t_1 s_1^{-a_1} s_2^{-a_2} \ldots s_p^{-a_p} = s_1^{a_1 - a_2} s_2^{a_2 - a_3} \ldots s_p^{a_p - a_1} t_1$$

will be found in $\{H, t_1\}$ only when what precedes t_1 is a substitution in H. Since the difference between any two different a's is less than i_1—the lowest exponent of s_i in G_{t, i_1} besides zero—it follows that we must have

$$a_1 = a_2 = \ldots = a_p = \beta,$$

where $\beta = 0, 1, 2, \ldots$, or $i_1 - 1$. Hence we need consider only the i_1 substitutions of the form

$$(s_1 s_2 \ldots s_p)^\beta t_2.$$

If any of these i_1 substitutions generates with $\{H, t_1\}$ a group that corresponds to P_{i_2} $(i_2 < p - 1)$, its $\frac{p-1}{i_2}$th power must be in the head. This will be true only when the corresponding β satisfies one of the congruences

$$\beta\left(\frac{p-1}{i_2}\right) \equiv h i_1 \qquad (\text{mod } q - 1), \tag{1}$$

where h is any integer. Suppose that β_1 and β_2 are any two distinct values of

β, each of which satisfies one of these congruences. Each gives rise to a group that corresponds to P_{i_2} $(i_2 < p - 1)$. Further, the resulting groups are distinct. If one could be transformed into the other by a substitution (s), then s would have to transform H into itself, since H is the only invariant intransitive subgroup of order $\left[\frac{q(q-1)}{i_1}\right]^p$ that is contained in either. It follows, therefore, that s must transform the division containing $(s_1 s_2 \ldots . s_p)^{\beta_1} t_2$ into that containing $(s_1 s_2 \ldots . s_p)^{\beta_2} t_2$, the divisions being formed with respect to $\{H, t_1\}$. This clearly cannot be done. Hence the number of imprimitive groups that correspond to P_{i_2} $(i_2 < p - 1)$ is equal to the number of values of β (where $\beta = 0, 1, \ldots . , i_1 - 1$) that satisfy the congruences (1).

9. THEOREM.—*The number of imprimitive groups of degree* pq *that contain the head*

$$H \equiv \{G_{1, i_1}, G_{2, i_1}, \ldots . , G_{p, i_1}\}_{1, 1, \ldots . , 1}$$

and whose substitutions interchange the systems of intransitivity of H *according to* P_{i_2} *is as follows when* $p \neq q$:

When $i_2 = p - 1$ *there are two groups if* i_1 *is a multiple of* p, *but just one group if* i_1 *is not a multiple of* p.

When $i_2 < p - 1$, *the number of groups is equal to the number of solutions of the congruences*

$$\beta\left(\frac{p-1}{i_2}\right) \equiv h i_1 \qquad (\text{mod } q - 1),$$

where h *is any integer and* β *is restricted to the values* $0, 1, 2, \ldots . , i_1 - 1$.

The head is formed by writing after each substitution in G_{1, i_1} the same substitution in the other sets of letters.

I. Suppose that $i_1 < q - 1$.

In this case the largest group within which H is invariant without having its systems interchanged is $\{G_1, G_2, \ldots . , G_p\}_{1, 1, \ldots . , 1}$. This is of order $q(q-1)$ while H is of order $q(q-1)/i_1$. There are then i_1 sets of substitutions that transform according to any substitution in P_{i_2}. Since $(s_1 s_2 \ldots . s_p)^{i_1}$ is the lowest power of $s_1 s_2 \ldots . s_p$ that occurs in H, it is clear that the i_1 sets of substitu-

tions that transform according to p_1 may be obtained by multiplying the substitutions in the head by

$$(s_1 s_2 \ldots\ldots s_p)^a t_1, \tag{A}$$

where $a = 0, 1, 2, \ldots\ldots, i_1 - 1$. The p^{th} power of each of these substitutions that generates with H a group corresponding to P_{p-1}, must be in H. This will be true only when a satisfies one of the congruences

$$ap \equiv hi_1 \qquad (\text{mod } q - 1), \tag{1}$$

where h is any integer. These congruences may be written in the form

$$\begin{aligned} ap &= hi_1 + k(q-1), \\ &= (h + kM)\, i_1, \end{aligned}$$

where k is any integer and M equals $(q-1)/i_1$. Hence

$$a = \frac{h + kM}{p}\, i_1.$$

From this equation it follows that zero is the only value of a less than i_1 that satisfies the congruence (1) unless i_1 is a multiple of p. If, however,

$$i_1/p = m,$$

m being an integer, then a may have the values $0, m, 2m, \ldots\ldots, (p-1)\,m$. Hence when i_1 is a multiple of p, the substitutions (A) may be replaced by the substitutions

$$(s_1 s_2 \ldots\ldots s_p)^{mx}\, t_1,$$

where $x = 0, 1, 2, \ldots\ldots, p-1$. Each of these substitutions generates with the head a group that corresponds to P_{p-1}, but it can be proved that only two of them are distinct. In the group that contains the substitution $(s_1 s_2 \ldots\ldots s_p)^m\, t$ are found the substitutions $(s_1 s_2 \ldots\ldots s_p)^{my}\, t_1^y$, where y may have any of the values $1, 2, , \ldots\ldots, p-1$. If now p_2 be of order $p-1$, then a suitable power of the substitution t_2 that permutes according to p_2 will transform the head into itself and the substitution $(s_1 s_2 \ldots\ldots s_p)^{mx}\, t_1$ into $(s_1 s_2 \ldots\ldots s_p)^{mx}\, t_1^x$ if x is different from zero. Hence the $p-1$ groups $\{H, (s_1 s_2 \ldots\ldots s_p)^{mx}\, t_1\}$, when x has the values $1, 2, \ldots\ldots, p-1$, are conjugate. The groups $\{H, t_1\}$ and $\{H, (s_1 s_2 \ldots\ldots s_p)^m\, t_1\}$ are distinct abstract groups, since one contains an invariant operator of order p while the other does not. Hence it is proved that when i_1 is divisible by p, there are two groups containing the given head that

correspond to P_{p-1}, while when i_1 is not divisible by p, there is just one such group.

The i_1 sets of substitutions that transform according to p_2 may be obtained by multiplying the substitutions in the head by

$$(s_1 s_2 \ldots\ldots s_p)^\beta t_2,$$

where β may take the values $0, 1, \ldots\ldots, i_1 - 1$. Each of these transforms both the head and $\{H, t_1\}$ into themselves. The $\frac{p-1}{i_2}$th power of $(s_1 s_2 \ldots\ldots s_p)^\beta t_2$ will be found in the head only when β satisfies one of the relations

$$\beta\left(\frac{p-1}{i_2}\right) \equiv h i_1 \qquad (\text{mod } q-1), \tag{2}$$

where h is any integer. Each value of β that satisfies one of these congruences gives then a substitution which generates with $\{H, t_1\}$ a group that corresponds to $P_{i_2}(i_2 < p - 1)$. And by the argumeut used in the preceding theorem, it follows that the groups thus obtained are all distinct. Hence the number of groups that contain $\{H, t_1\}$ as an invariant subgroup and that correspond to $P_{i_2}(i_2 < p - 1)$ is equal to the number of values of β that satisfy the congruences (2), β being restricted to the numbers $0, 1, \ldots\ldots, i_1 - 1$.

If $\{H, (ss \ldots\ldots s)^m t_1\}$ is contained as an invariant subgronp of a group that corresponds to P_{i_2}, then

$$(s_1 s_2 \ldots\ldots s_p)^\beta t_2 (s_1 s_2 \ldots\ldots s_p)^m t_1 t_2^{-1} (s_1 s_2 \ldots\ldots s_p)^{-\beta} = (s_1 s_2 \ldots\ldots s_p)^m t_1^\gamma$$

must be a substitution in $\{H, (s_1 s_2 \ldots\ldots s_p)^m t_1\}$; γ is defined by the relation $t_2 t_1 t_2^{-1} = t_1^\gamma$. If this be true, it must occur in the division in which $[(s_1 s_2 \ldots\ldots s_p)^m t_1]^\gamma$, or $(s_1 s_2 \ldots\ldots s_p)^{m\gamma} t_1^\gamma$ is found. In this division $s_1 s_2 \ldots\ldots s_p$ occurs to the powers $m\gamma + h i_1$. Hence there must be a relation of the form

$$m\gamma + h i_1 \equiv m \quad (\text{mod } q - 1),$$

$$\therefore \qquad m(\gamma - 1) = (kM - h)\, i_1, \ (\text{where } q - 1 = M i_1),$$

$$\therefore \qquad m = \frac{i_1}{p} = \frac{kM - h}{\gamma - 1}\, i_1,$$

or

$$\gamma - 1 \quad = p\,(kM - h).$$

This equation however cannot exist, since γ is not congruent to unity modulus p. Hence there is no group that contains $\{H, (S_1 S_2 \ldots\ldots S_p)^m t_1\}$ as an invariant subgroup and that corresponds to P_{i_2}, i_2 being different from $p - 1$.

II. Suppose that $i_1 = q - 1$.

In this case the largest group within which the head is invariant without having its systems interchanged is $\{G_1, G_2, \ldots, G_p\}_{q,q,\ldots,q}$. This latter group is formed by multiplying the corresponding divisions with respect to the self-conjugate subgroup of order q in the different sets of elements.

The groups that contain the given head and are isomorphic to P_{p-1} are of order and degree pq. We know, then, that when p is not a divisor of $q-1$, there is just one such imprimitive group, and that when p is a divisor of $q-1$, there are two such groups. These two groups may be taken as $\{H, t_1\}$ and $\{H, (ss \ldots s)^m t_1\}$, where $m = (q-1)/p$,

The most general substitution that permutes according to p_2 is of the form

$$S_1^{\alpha_1} S_2^{\alpha_2} \ldots S_p^{\alpha_p} (s_1 s_2 \ldots s_p)^\beta t_2,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p = 0, 1, \ldots, q-1$ and $\beta = 0, 1, \ldots, q-2$. If this substitution transforms $\{H, t_1\}$ into itself, then $S_1^{\alpha_1} S_2^{\alpha_2} \ldots S_p^{\alpha_p}$ must also do so. Now

$$S_1^{\alpha_1} S_2^{\alpha_2} \ldots S_p^{\alpha_p} t_1 S_1^{-\alpha_1} S_2^{-\alpha_2} \ldots S_p^{-\alpha_p} = S_1^{\alpha_1 - \alpha_2} S_2^{\alpha_2 - \alpha_3} \ldots S_p^{\alpha_p - \alpha_1} t_1.$$

If this be in $\{H, t_1\}$, it must be of the form

$$(S_1 S_2 \ldots S_p)^\alpha t_1.$$

That is, we must have the relations

$$\alpha_i - \alpha_1 \equiv -(i-1)\alpha \qquad (\text{mod } q),$$

where $i = 1, 2, \ldots, p$ and α has one of the values $0, 1, \ldots, q-1$. Such can be the case only when

$$\alpha p \equiv 0 \qquad (\text{mod } q).$$

Zero is the only value of α less than q that satisfies this congruence, and hence we must have $\alpha_1 = \alpha_2 = \ldots = \alpha_p$. The substitutions that permute according to p_2 which are to be considered, may therefore be obtained by multiplying the substitutions of the head by the substitutions

$$(s_1 s_2 \ldots s_p)^\beta t_2,$$

where $\beta = 0, 1, 2, \ldots, q-1$. If the $\frac{p-1}{i_2}$th power of any of these be in the head, the corresponding β must satisfy the congruence

$$\beta\left(\frac{p-1}{i_2}\right) \equiv 0 \qquad (\text{mod } q-1).$$

To each such β there corresponds a distinct group, containing $\{H, t_1\}$ self-conjugately; hence formula (2) is true for all values of i_1.

It remains to find if $\{H, (s_1 s_2 \ldots . s_p)^m t_1\}$ is contained as an invariant subgroup in a group that corresponds to P_{i_2}, where i_2 is less than $p-1$. If we transform $(s_1 s_2 \ldots . s_p)^m t_1$ by the inverse of the general substitution

$$s_1^{a_1} s_2^{a_2} \ldots . s_p^{a_p} (s_1 s_2 \ldots . s_p)^\beta t_2$$

we get a substitution of the form

$$s_1^{a'_1} s_2^{a'_2} \ldots . s_p^{a'_p} (s_1 s_2 \ldots . s_p)^m t_1^\gamma .$$

This could not occur in $\{H, (s_1 s_2 \ldots . s_p)^m t_1\}$; for it would have to be found in the division in which $(s_1 s_2 \ldots . s_p)^{m\gamma} t_1^\gamma$ occurs and we would then have the relation

$$m\gamma \equiv m \qquad (\text{mod } q-1),$$

or

$$m(\gamma - 1) = k(q-1).$$

or

$$\gamma - 1 = k \frac{q-1}{m} = kp.$$

This cannot be true since γ is not congruent to unity modulus p. Hence the given group is not contained self-conjugately in a group that corresponds to P_{i_2}, where $i_2 < p - 1$.

When $p = q$ and $i_1 < p - 1$, the above argument shows that there is just one group containing H that corresponds to P_{p-1}; and that when $i_2 < p - 1$, the number that corresponds to P_{i_2} is not greater than the number of solutions of the congruences

$$\beta \left(\frac{p-1}{i_2}\right) \equiv hi_1 \qquad (\text{mod } p-1),$$

where β is restricted to the values 0, 1, 2, $\ldots .$, $i_1 - 1$ and h is any integer.

When $p = q$ and $i_1 = p - 1$, the groups that correspond to P_{i_2} are easily determined by considering the holomorphs of the two regular groups of order p^2. In the holomorph of the cyclic group of order p^2 there is clearly just one group that contains the given head and that corresponds to P_{i_2}.

10. THEOREM.—*When* $i_2 = p - 1$ *and* $\frac{q-1}{i_1}$ *is not a multiple of* p, *there is one imprimitive group of degree pq that contains the head*

$$H \equiv \{G_{1, i_1}, G_{2, i_1}, \ldots . , G_{p, i_1}\}_{q, q, \ldots ., q},$$

whose substitutions interchange the systems of intransitivity of H according to P_{i_2}; when $i_2 = p - 1$ and $\frac{q-1}{i_1}$ is a multiple of p, there are two such groups.

When $i_2 < p - 1$, the number of groups is equal to the number of solutions of the congruences

$$\beta\left(\frac{p-1}{i_2}\right) \equiv hi_1 \qquad (\mathrm{mod}\ q-1),$$

where h is any integer and β is restricted to the values $0, 1, 2, \ldots, i_1 - 1$.

The given head in this case is formed by taking the product of corresponding divisions with respect to the invariant subgroup q of each of the transitive constituents. The largest group within which H is invariant without having its systems interchanged is $\{G_1, G_2, \ldots, G_p\}$. Hence we may take

$$s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_p^{\alpha} t_1,$$

$\alpha_1, \alpha_2, \ldots, \alpha_p$ being integers, as the general substitution that transforms according to p_1. This substitution transforms the head into itself and with H generates a group that corresponds to P_{p-1}, in case its p^{th} power is found in the head. This will be true if the α's satisfy one of the relations

$$\alpha_1 + \alpha_2 + \ldots + \alpha_p \equiv hi, \qquad (\mathrm{mod}\ q-1), \tag{1}$$

h being any integer.

We first show that the groups that correspond to different solutions of any given one of these $\frac{q-1}{i_1}$ congruences—this being the number of distinct relations (1)—are conjugate. For, let h_1 be any value of h, and let $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\alpha_1', \alpha_2', \ldots, \alpha_p'$ be two distinct solutions of the corresponding congruence. Then

$$s_1^{\beta_1} s_2^{\beta_2} \ldots s_p^{\beta_p} s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_p^{\alpha_p} t_1 s_1^{-\beta_1} s_2^{-\beta_2} \ldots s_p^{-\beta_p}$$
$$= s_1^{\beta_1 - \beta_2 + \alpha_1} s_2^{\beta_2 - \beta_3 + \alpha_2} \ldots s_p^{\beta_p - \beta_1 + \alpha_p} t_1 = s_1^{\alpha_1'} s_2^{\alpha_2'} \ldots s_p^{\alpha_p'} t_1,$$

when $\beta_i = \beta_1 + (\alpha_1 + \alpha_2 + \ldots + \alpha_{i-1}) - (\alpha_1' + \alpha_2' + \ldots + \alpha_{i-1}')$ where $i = 2, 3, \ldots, p$. Also $s_1^{\beta_1} s_2^{\beta_2} \ldots s_p^{\beta_p}$ transforms the head into itself. Hence there cannot be more than $\frac{q-1}{i_1}$ distinct groups that correspond to P_{p-1}— one to each of the above congruences.

The substitutions in the following table give a set of $q-1$ generators—one corresponding to each congruence—which, with the given head, generate one set of $\frac{q-1}{i_1}$ groups isomorphic to P_{p-1}:

$$\begin{array}{cccc}
t_1, & s_1^{i_1} t_1, & & s_1^{(p-1)i_1} t_1, \\
(s_1 s_2 \ldots\ldots s_p)^{i_1} t_1, & s_1^{i_1} (s_1 s_2 \ldots\ldots s_p)^{i_1} t_1, & \ldots\ldots, & s_1^{(p-1)i_1} (s_1 s_2 \ldots\ldots s_p)^{i_1} t_1, \\
(s_1 s_2 \ldots\ldots s_p)^{2i_1} t_1, & s_1^{i_1} (s_1 s_2 \ldots\ldots s_p)^{2i_1} t_1, & \ldots\ldots, & s_1^{(p-1)i_1} (s_1 s_2 \ldots\ldots s_p)^{2i_1} t_1, \\
\vdots & \vdots & \ldots\ldots & \vdots
\end{array}$$

the table being continued until $\frac{q-1}{i_1}$ substitutions are contained in it.

In the first place it is evident that the groups that correspond to the substitutions in the i^{th} column where $i = 1, 2, \ldots\ldots, p$, are identical. Hence we need consider only those groups that correspond to the substitutions in the first row. In the group $\{H, s_1^{i_1} t_1\}$ are found the substitutions

$$(s_1 s_p s_{p-1} \ldots\ldots s_{p-x+2})^{i_1} t_1,$$

where $x = 2, 3, \ldots\ldots, p-1$. If p_2 be of order $p-1$, then a certain power of the substitution t_2 that transforms according to p_2 will transform t_1^x into t_1 and the group $\{H, s_1^{1} t_1\}$ into a group that is conjugate to $\{H, s_1^{xi_1} t_1\}$. Hence the groups $\{H, s_1^{ri_1} t_1\}$, where $r = 1, 2, \ldots\ldots, p-1$ are conjugate. We consider then the two groups $\{H, t_1\}$ and $\{H, s_1^{i_1} t_1\}$. The latter contains in the division in which $s_1^{i_1} t_1$ occurs the substitutions

$$S_1^{\alpha_1'} S_2^{\alpha_2'} \ldots\ldots S_p^{\alpha_p'} S_1^{hi_1 + t_1} (s_2 s_3 \ldots\ldots s_p)^{hi_1} t_1,$$

where $\alpha_1', \alpha_2', \ldots\ldots, \alpha_p' = 0, 1, \ldots\ldots$, or $q-1$ and h is any integer. If the p^{th} power of any of these be identity, then

$$(hp + 1)\, i_1 \equiv 0 \qquad (\text{mod } q-1),$$

or

$$(hp + 1)\, i_1 = k(q-1),$$

or

$$kM - hp = 1, \tag{2}$$

if $M = (q-1)/i_1$. When M is not a multiple of p, this equation has a solution. In this case there is a substitution of order p in the division in question. The corresponding exponents of the s's then satisfy the first congruence of (1), and the group is conjugate to $\{H, t_1\}$. When, however, $\frac{q-1}{i_1} = mp$, where m is an

integer, then (2) becomes

$$(km - h)\,p = 1\,.$$

This has no solution. Hence when $(q-1)/i_1$ is not a multiple of p, there is just one group containing the given head that corresponds to P_{p-1}; but when $(q-1)/i_1$ is a multiple of p, there are two such groups.

The general form of the generating substitutions that transform according to p_2 is

$$s_1^{a_1}s_2^{a_2}\;\ldots\ldots\;s_p^{a_p}\,t_2\,.$$

If any of these transform $\{H, t_1\}$ into itself, then $s_1^{a_1}s_2^{a_2}\;\ldots\ldots\;s_p^{a_p}$ must do so. That is,

$$s_1^{a_1}s_2^{a_2}\;\ldots\ldots\;s_p^{a_p}\,t_1\,s_1^{-a_1}\,s_2^{-a_2}\;\ldots\ldots\;s_2^{-a_p} = s_1^{a_1-a_2}\,s_2^{a_2-a_3}\;\ldots\ldots\;s_p^{a_p-a_1}\,t_1$$

must be of the form

$$(s_1s_2\;\ldots\ldots\;s_p)^{a}\,t_1,$$

where α is some multiple of i_1. That is, we have the relations

$$\alpha_i \equiv \alpha_1 - (i-1) \qquad (\text{mod } q-1).$$

Putting $i = p$, it is seen that

$$\alpha p \equiv 0 \qquad (\text{mod } q-1),$$

i. e.,

$$\alpha = k\,\frac{q-1}{p}\,.$$

Hence, as the values of α to be considered are multiples of i_1 that are less than $q-1$, it follows that $q-1$ must be a multiple of pi_1 unless $\alpha = 0$. In the latter case, $\alpha_1 = \alpha_2 = \;\ldots\ldots\; = \alpha_p$. That is, when $(q-1)/i_1$ is not a multiple of p, the substitutions that permute according to p which we have to consider may be obtained by multiplying the head by the substitutions

$$(s_1s_2\;\ldots\ldots\;s_p)^{\beta}\,t_2\,,$$

where $\beta = 0, 1, 2, \ldots\ldots i_1 - 1$. When, however, $(q-1)/i_1 = mp$, we must multiply the head by

$$s_1^{a_1}\,s_2^{a_1-a}\,s_3^{a_1-2a}\;\ldots\ldots\;s_p^{a_1-(p-1)a}\,t_2\,,$$

where $\alpha = 0, mi_1, 2mi_1, \ldots\ldots, (p-1)\,mi_1$. These may be written in the form

$$s_2^{(p-1)a}s_3^{(p-2)a}\;\ldots\ldots\;s_p^{a}\,(s_1\,s_2\ldots\ldots s_p)^{a_1}\,t_2.$$

If α_1' is a value of α_1 for which the $(p-1)/i_2^{\text{th}}$ power of this substitution is in the head, then there cannot be more than p groups that correspond to this value of α_1. We prove that there is just one—i. e., that the groups that correspond to the substitutions

$$s_2^{(p-1)\alpha} s_3^{(p-2)\alpha} \ldots\ldots s_p^{\alpha} (s_1 s_2 \ldots\ldots s_p)^{\alpha_1} t_2$$

are conjugate. For, transform the substitution $(s_1 s_2 \ldots\ldots s_p)^{\alpha_1'} t_2$ by the inverse of $s_2^{(p-1)\alpha} s_3^{(p-2)\alpha} \ldots\ldots s_p^{\alpha}$. This latter transforms $\{H, t_1\}$ into itself. Suppose that t_2^{-1} transforms S_x into S_2. Then the transform in question is of the form

$$s_2^{(x-2)\alpha} \ldots\ldots (s_1 s_2 \ldots\ldots s_p)^{\alpha_1'} t_2.$$

When α takes the values $0, mi_1, \ldots\ldots, (p-1)\, mi_1$, we get p different substitutions in this way. For, let imi_1 and jmi_1 be any two distinct values of α. If these gave rise to the same substitution, then we would have

$$(x-2)\, imi_1 \equiv (x-2) jmi_1 \qquad (\text{mod } q-1),$$

or $$(x-2)(i-j)\, mi_1 = k(q-1) = kmpi_1,$$

or $$(x-2)(i-j) = kp.$$

$\therefore$ $$x - 2 \equiv 0 \qquad (\text{mod } p),$$

$$x \equiv 2 \qquad (\text{mod } p).$$

This, however, is not true. Hence, in any case, the substitutions that transform according to p_2 may be found by multiplying H by the substitutions

$$(s_1 s_2 \ldots\ldots s_p)_1^{\beta} t_2,$$

where $\beta = 0, 1, \ldots\ldots, i_1 - 1$. As above (§8), we prove that to each value of β that satisfies the congruences

$$\beta\left(\frac{p-1}{i_2}\right) \equiv hi_1 \qquad (\text{mod } q-1),$$

there is a distinct group that contains $\{H, t_1\}$ as an invariant subgroup.

It remains to consider what (if any) groups contain $\{H, S_1^{i_1} t_1\}$ as an invariant subgroup which corresponds to P_{i_2}. The general substitution to be considered is, as before,

$$s_1^{\alpha_1} s_2^{\alpha_2} \ldots\ldots s_p^{\alpha_p}\; t_2.$$

Transforming $s_1^{i_1} t_1$ by this, we get

$$t_2^{-1} s_1^{-\alpha_1} \ldots\ldots s_p^{-\alpha_p} s_1^{i_1} t_1 s_1^{\alpha_1} \ldots\ldots s_p^{\alpha_p} t_2 = t_2^{-1} s_1^{\alpha_2 - \alpha_1 + i_1} s_2^{\alpha_3 - \alpha_2} \ldots\ldots s_p^{\alpha_1 - \alpha_p} t_1 t_2$$
$$= s_1^{\alpha_2 - \alpha_1 + i_1} s_{i_2}^{\alpha_3 - \alpha_2} s_{i_3}^{\alpha_4 - \alpha_3} \ldots\ldots s_{i_p}^{\alpha_1 - \alpha}\; t_1^{\gamma'}, \qquad (\text{A}_1)$$

where the subscripts $i_2, i_3, \ldots\ldots i_p$ denote the numbers $2, 3, \ldots\ldots, p$ in some order and where γ' is defined by $t_2^{-1} t_1 t_2 = t_1^{\gamma'}$. If this be a substitution in $\{H, s_1^{i_1} t_1\}$, it must occur in the division in which $(s_1^{i_1} t_1)^{\gamma'}$ or $(s_1 s_p \ldots\ldots s_{p-\gamma'+2})^{i_1} t_1^{\gamma'}$ is found. In this division we have the substitutions

$$(s_1 s_2 \ldots\ldots s_p)^{hi_1}(s_1 s_p \ldots\ldots s_{p-\gamma'+2})^{i_1} t_1^{\gamma'}. \qquad (\mathrm{A}_2)$$

If (A_1) be identical with any of these, then the p^{th} power of each must give rise to the same substitution. Now the p^{th} power of (A_1) is $(s_1 s_2 \ldots\ldots s_p)^{i_1}$, while that of (A_2) is $(s_1 s_2 \ldots\ldots s_p)^{(hp+\gamma')i_1}$. Hence we must have the relation

$$(hp + \gamma')\, i_1 \equiv i_1 \qquad (\mathrm{mod}\ q - 1),$$

or

$$(hp + \gamma')\, i_1 = (1 + kmp)\, i_1, \qquad \left\{\text{since } \frac{q-1}{i_1} = mp\right\},$$

or

$$\gamma' = 1 + (km - h)\, p.$$

This, however, cannot be true, since γ' is not congruent to unity modulus p. Hence there is no group that contains $\{H, s_1^{i_1} t_1\}$ as an invariant subgroup and that corresponds to P_{i_2}, i_2 being different from $p - 1$.

11. Let H_{i_1} denote that invariant subgroup of

$$(S_1 S_2^{-1})(S_2 S_3^{-1}) \ldots\ldots (S_{p-1} S_p^{-1})(s_1 s_2 \ldots\ldots s_p)$$

whose index under G is i_1, p being an odd prime.

THEOREM.—*The number of imprimitive groups of degree pq that contain the head*

$$H_{i_1}$$

and whose substitutions interchange the systems of intransitivity of H_{i_1} according to P_{i_2} is as follows:

(a) *When $i_2 = p - 1$ and $p = q$ there are two groups if $i_1 = q - 1$ and one group if $i_1 < q - 1$.*

(b) *When $i_2 = p - 1$ and $p \neq q$ there is one group if i_1 is not a multiple of p and two groups if i_1 is a multiple of p.*

(c) *When $i_2 < p - 1$, $p \neq q$, and $i_1 = q - 1$ the number of groups is equal to the number of solutions of the congruences*

$$\beta \left(\frac{p-1}{i}\right) \equiv h i_1 \qquad (\mathrm{mod}\ q - 1), \qquad (1)$$

where h is any integer and $\beta = 0, 1, \ldots, i_1 - 1$ *in case* $\frac{p-1}{i_2}$ *is not a mulliple of* q *and one greater than this number in case* $\frac{p-1}{i_2}$ *is a multiple of* q; *when* $i_2 < p - 1$, $p \neq q$, $i_1 < q - 1$ *the number is given by* (1).

(d) *When* $i_2 < p - 1$ *and* $p = q$ *the number is one greater than that given by the congruences* (1).

Suppose $i_1 = q - 1$.

In this case the largest group within which the given head is invariant without having its systems interchanged is $(S_1) \ldots (S_p)(s_1 s_2 \ldots s_p)$. This is of order $q^p(q-1)$, while H_{i_1} is of order q^{p-1}. There are then $q(q-1)$ sets of substitutions that transform according to any substitution in P_{i_2}. Now, H_{q-1} contains the substitutions

$$(S_1 S_2 \ldots S_{p-1})^a S_p^{a-pa},$$

where a is any integer. This substitution will be of the form $(S_1 S_2 \ldots S_p)^a$ when

$$ap \equiv 0 \qquad (\bmod q).$$

It follows that when p is different from q, there is no substitution of the form $(S_1 S_2 \ldots S_p)^a$ in H_{q-1}. In this case the substitutions that permute according to p_1 may be obtained by forming the products of the head and the following substitutions:

$$\begin{array}{llll}
 & t_1, & s_1 s_2 \ldots s_p t_1, \ldots & (s_1 s_2 \ldots s_p)^{q-2} t_1 \\
(S_1 \ldots S_p) & t_1, & (S_1 \ldots S_p)\, s_1 s_2 \ldots s_p t_1, \ldots & (S_1 \ldots S_p)(s_1 s_2 \ldots s_p)^{q-2} t_1 \\
(S_1 \ldots S_p)^2 & t_1, & (S_1 \ldots S_p)^2 s_1 s_2 \ldots s_p t_1, \ldots & (S_1 \ldots S_p)^2 (s_1 s_2 \ldots s_p)^{q-2} t_1 \\
\vdots & & \vdots \quad \ldots & \vdots \\
(S_1 \ldots S_p)^{q-1} & t_1, & (S_1 \ldots S_p)^{q-1} s_1 s_2 \ldots s_p t_1, \ldots & (S_1 \ldots S_p)^{q-1}(s_1 s_2 \ldots s_p)^{q-2} t_1
\end{array}$$

Of the substitutions in the first column t_1 is the only one whose p^{th} power is in the head. If the p^{th} power of any other one in the first row is in the head then this is true of all the substitutions in the corresponding column. But the resulting groups are conjugate, as is seen by transforming the one containing $(s_1 s_2 \ldots s_p)^\beta t_1$ by $(S_1 S_2 \ldots S_p)^a$ where β is the exponent of $s_1 s_2 \ldots s_p$ in the column in question and $a = 1, 2, \ldots, q-1$. We need consider then only the

substitutions in the first row. The p^{th} power of $(s_1 s_2 \ldots . s_p)^a t_1$ will be in the head if

$$ap \equiv 0 \qquad (\text{mod } q-1).$$

When p is prime to $q-1$, zero is the only value that a may take and in this case there is just one group. When, however, $(q-1)/p$ equals an integer, m, then $a = 0, m, 2m, \ldots., (p-1)m$. But as above (§9) the groups that correspond to the values $m, 2m, \ldots., (p-1)m$ are conjugate and hence in this case there are two distinct groups that contain the given head and that correspond to P_{p-1}.

When p equals q the head contains the substitutions $(S_1 S_2 \ldots . S_p)^a$ where $a = 1, 2, \ldots., q-1$. The substitutions that permute according to p_1 may now be found by multiplying the head by the set of substitutions obtained by replacing $(S_1 S_2 \ldots . S_p)^a$ by S_1^a in the preceding table. Of these only those in the first column have their p^{th} power in the head. That is, we need consider only those of the form

$$S_1^a t_1$$

where $a = 0, 1, \ldots., p-1$. If $\{H, S_1 t_1\}$ is transformed by $(s_1 s_2 \ldots . s_p)^\beta$ where $\beta = 0, 1, \ldots., p-2$, it is seen that the groups $\{H, S_1^x t_1\}$ where $x = 1, 2, \ldots., p-1$ are conjugate. The groups $\{H, t_1\}$ and $\{H, S_1 t_1\}$ are distinct, since the former contains only substitutions of order p in the division in which t_1 occurs while the latter contains substitutions of order p^2 in this division.

When p is not equal to q, the substitutions that transform according to p_2 can be found by multiplying the head by the substitutions that result when t_1 is replaced by t_2 in the first of the above tables. If any substitution in the first column besides t_1 has its $(p-1)/i_2{}^{\text{th}}$ power in the head then $\frac{p-1}{i_2}$ must be a multiple of q. The groups $\{H, t_1, (S_1 S_2 \ldots . S_p)^x t_2\}$, where $x = 1, 2, \ldots., q-1$ are conjugate as is seen by transforming one of them by $(s_1 s_2 \ldots . s_p)^\beta$ where $\beta = 1, 2, \ldots., q-2$. But the groups $\{H, t_1, t_2\}$ and $\{H, t_1, (S_1 S_2 \ldots . S_p)\, t_2\}$ are clearly distinct. If any other substitution in the first row, besides t_2, has its $\frac{p-1}{i_2}$th power in the head, then all of those in the column to which it belongs also satisfy this condition. But the corresponding groups are conjugate, as is seen

on transforming one of them by $(S_1 S_2 \dots S_p)^\alpha$ where $\alpha = 1, 2, \dots, q-1$. We need then to consider only substitutions of the form

$$(s_1 s_2 \dots s_p)^\beta t_2$$

where $\beta = 0, 1, \dots, q-2$. As before, it follows that the number of distinct groups is equal to the number of values of β that satisfy the congruences

$$\beta \left(\frac{p-1}{i_2}\right) \equiv 0 \qquad (\text{mod } q-1).$$

If $\{H_{q-1}, (s_1 s_2 \dots s_p)^m t_1\}$ is contained as an invariant subgroup in a group that corresponds to P_{i_2} $(i_2 < p-1)$ then

$$t_2^{-1} (s_1 s_2 \dots s_p)^{-\beta} (S_1 S_2 \dots S_p)^{-\alpha} (s_1 s_2 \dots s_p)^m t_1 (S_1 S_2 \dots S_p)^\alpha (s_1 s_2 \dots s_p)^\beta t_2$$

or

$$(S_1 S_2 \dots S_p)^{\alpha'} \quad (s_1 s_2 \dots s_p)^m t_1^{\gamma_1}$$

must be a substitution of this group. This can be true only when it is identical with the substitution $(s_1 s_2 \dots s_p)^{m\gamma_1} t_1^{\gamma_1}$ — that is, when

$$m\gamma_1 \equiv m \qquad (\text{mod } q-1)$$

or

$$\gamma_1 \equiv 1 \qquad (\text{mod } p).$$

This equation, however, is not true, and hence there is no group of the kind in question.

When p equals q, we may replace t_1 by t_2 in the second of the above tables to obtain a set of substitutions which with H_{p-1} generate the substitutions that permute according to p_2. Those in the first column are of the form

$$S_1^\alpha t_2$$

where $\alpha = 0, 1, \dots, p-1$. The $\frac{p-1}{i_2}$th power of this is in the head when

$$\alpha \left(\frac{p-1}{i_2}\right) \equiv 0 \qquad (\text{mod } p)$$

or

$$\alpha = \frac{kpi_2}{p-1}.$$

It follows that zero is the only permissible value of α. If any substitution in the first row has its $\frac{p-1}{i_2}$th power in the head, then the corresponding β must satisfy the relation

$$\beta \left(\frac{p-1}{i_2}\right) \equiv 0 \qquad (\text{mod } p-1), \tag{1}$$

where $\beta = 0, 1, 2, \ldots, p-2$. If β_1 be any solution of this congruence besides zero, then it is clear that all the substitutions in the same column with $(s_1 s_2 \ldots s_p)^{\beta_1} t_2$ have their $\frac{p-1}{i_2}$th power in the head. Now all of these substitutions that have their $\frac{p-1}{i_2}$th power in the head also transform $\{H_{p-1}, t_1\}$ into itself. But those which belong to the same column give rise to conjugate groups as is seen by transforming by S_1^x where $x = 1, 2, \ldots, p-1$. Hence the number of distinct groups that contain $\{H_{p-1}, t_1\}$ as an invariant subgroup is equal to the number of solutions of the congruence (1).

If $(s_1 s_2 \ldots s_p)^{\beta_1} t_2$ transforms $\{H_{p-1} S_1 t_1\}$ into itself, so also will $S_1^{a_1}(s_1 s_2 \ldots s_p)^{\beta_1} t_2$. We need consider then only those substitutions in the first row whose exponents satisfy (1). Now if β_1 be such a value of the exponent, then

$$t_2^{-1}(s_1 s_2 \ldots s_p)^{-\beta_1} S_1 t_1 (s_1 s_2 \ldots s_p)^{\beta_1} t_2 = s_1^{-\beta_1} S_1 s_1^{\beta_1} t_1^{\gamma_1} = S_1^{\alpha'} t_1^{\gamma_1},$$

where $t_2^{-1} t_1 t_2 = t_1^{\gamma_1}$ and $s_1^{-\beta_1} S_1 s_1^{\beta} = S_1^{\alpha'}$. This will evidently be a substitution in $\{H_{p-1}, S_1 t_1\}$ only when $\alpha' = \gamma_1$. Hence, there is just one substitution in the first row that transforms the given group into itself—the corresponding value of β being i_2. All the substitutions in the corresponding column give rise to conjugate groups, as may be seen by transforming $\{H, S_1 t_1, (s_1 s_2 \ldots s_p)^{i_2} t_2\}$ by S_1^{α}, where $\alpha = 1, 2, \ldots, p-1$. Hence, there is just one group that contains $\{H, S_1 t_1\}$ and that corresponds to P_{i_2}.

(2). When $i_1 < q-1$.

In this case, the largest group within which the given head is invariant without having its systems interchanged is

$$(S_1 S_2^{-1})(S_2 S_3^{-1}) \ldots (S_{p-1} S_p^{-1})(s_1 s_2 \ldots s_p).$$

There are then i_1 sets of substitutions that transform according to any substitution in P_{i_2}. Those that transform according to P_{p-1} may be obtained by multiplying the head by the substitutions

$$(s_1 s_2 \ldots s_p)^{\alpha} t_1,$$

where $\alpha = 0, 1, \ldots, i_1 - 1$. If the p^{th} power of any of these be in the head,

then the corresponding α must satisfy the relation

$$\alpha p \equiv h i_1 \qquad (\text{mod } q-1),$$

where h is as usual. That is,

$$\alpha p = (h + kM)\, i_1 \qquad \left\{\frac{q-1}{i_1} = M\right\}.$$

When i_1 is not a multiple of p, zero is the only value of α that can be used; when $i_1 = mp$ (where m is an integer), then $\alpha = 0, m, 2m, \ldots, (p-1)m$. Hence, as before, when p is prime to i_1, there is just one group that corresponds to P_{p-1}, and when $i_1 = mp$, there are two such groups.

The substitutions that transform according to p_2 result when the head is multiplied by the substitutions

$$(s_1 s_2 \ldots s_p)^\beta t_2,$$

where $\beta = 0, 1, \ldots, i_1 - 1$. Each of these transforms both the head and $\{H_{i_1}, t_1\}$ into themselves. The number of groups that contain $\{H_{i_1}, t_1\}$ self-conjugately and that correspond to P_{i_2} is then equal to the number of solutions of the congruences

$$\beta\left(\frac{p-1}{i_2}\right) \equiv h i_1 \qquad (\text{mod } q-1),$$

where h is as usual.

By the reasoning used above, it follows that there is no group isomorphic to P_{i_2} that contains $\{H_{i_1}, (s_1 s_2 \ldots s_p)^m t_1\}$ as an invariant subgroup, where $i_2 < p - 1$.

12. Consider the head which may be written in the form

$$H \equiv (s_1 s_2^{-1})(s_2 s_3^{-1})(s_3 s_4^{-1}) \ldots (s_{p-1} s_p^{-1}).$$

To form the group which this represents, we first establish the $q:q$ isomorphism between G_1 and G_2 in which the division containing s_2^{-1} in G_2 corresponds to that containing s_1 in G_1. Denote this group by h_1. Then establish the $q^2(q-1):q$ isomorphism between $\{h_1, s_2\}$ and G_3 in which the division containing s_3^{-1} in G_3 corresponds to that containing s_2 in the group $\{h_1, s_2\}$—the divisions in the latter group being formed with respect to h_1. Denote the resulting group by h_2 and establish a similar isomorphism between $\{h_2, s_3\}$ and G_4; that is, make the division containing s_4^{-1} in G_4 correspond to the division containing

s_3 in $\{h_2, s_3\}$—the divisions being formed with respect to h_2 in the latter group. Continue this process until all the p groups G_t have been used. The resulting group evidently permits a cyclic interchange of the systems, and hence may be used as a head. It further permits the interchange of the systems required by p_2, and we now consider the groups that contain this head and that correspond to P_{i_2}.

THEOREM.—*When* $i_2 = p - 1$ *and* $q - 1$ *is a multiple of* p, *there are two groups that contain the head* H, *whose substitutions interchange the systems of intransitivity of* H *according to* P_{i_2}; *when* $i_2 = p - 1$ *and* $q - 1$ *is prime to* p, *there is just one such group.*

When $i_2 < p - 1$, *the number of groups is equal to the number of solutions of the congruence*

$$\beta\left(\frac{p-1}{i_2}\right) \equiv 0 \qquad (\text{mod } q - 1),$$

where $\beta = 0, 1, 2, \ldots., q - 2$.

The largest group within which the given head is invariant without having its systems interchanged is $\{G_1, G_2, \ldots. G_p\}$. This is of order $q^p(q-1)^p$ while the head is of order $q^p(q-1)^{p-1}$. Hence there are $q - 1$ sets of substitutions that permute according to any substitution in P_{i_2}. Those that transform according to p_1 may be found by multiplying the head by the substitutions

$$s_1^a t_1, \tag{A}$$

where $a = 0, 1, \ldots., q - 2$. Each of these transforms the head into itself The p^{th} power of (A) will be in the head if a satisfies the congruence

$$ap \equiv 0 \qquad (\text{mod } q - 1). \tag{1}$$

For we find in the head the substitutions

$$s_1^a s_2^{-a}; \quad s_2^{2a} s_3^{-2a}; \quad s_3^{3a} s_4^{-3a}; \quad \ldots.; s_{p-1}^{(p-1)a} s_p^{-(p-1)a}.$$

The product of these is $(s_1 s_2 \ldots. s_{p-1})^a s_p^{-(p-1)a}$ and this will be identical with $(s_1 s_2 \ldots. s_p)^a$ only when

$$-(p-1)\alpha \equiv \alpha \qquad (\text{mod } q - 1)$$

or

$$ap \equiv 0 \qquad (\text{mod } q - 1).$$

When p is not a divisor of $q - 1$, zero is the only value of a less than $q - 1$ that satisfies the congruence (1) and so in this case there is just one group containing the given head that corresponds to P_{p-1}. When, however, $(q-1)/p$ equals an

integer m, then $\alpha = 0, m, 2m, \ldots . (p-1)m$ are solutions of (1). The substitutions (A) that need to be considered may therefore be written in the form

$$s_1^{mx} t_1,$$

where $x = 0, 1, \ldots . , p-1$. These may evidently be replaced by the substitutions

$$(s_1 s_p \ldots . s_{p-x+2})^m t_1,$$

where s_{p-x+2} stands for unity when $x = 0$ and for s_1 when $x = 1$. Now by the method used above (§10) it follows that the groups corresponding to the values $x = 1, 2, \ldots . , p-1$ are conjugate. Hence in this case there are two groups that correspond to P_{p-1}.

The $q-1$ sets of substitutions that transform according to p_2 result when the head is multiplied by the substitutions

$$s_1^\beta t_2, \qquad (\mathrm{A}_1)$$

where $\beta = 0, 1, \ldots . , q-2$. Each of these transforms the head and also $\{H, t_1\}$ into themselves. The $\frac{p-1}{i_2}$th power of (A_1) will be in the head if β satisfies the relation

$$\beta\left(\frac{p-1}{i}\right) \equiv 0 \quad (\mathrm{mod}\, q-1). \qquad (2)$$

To each value of β that satisfies this congruence there corresponds a distinct group that corresponds to P_{i_2} $(i_2 < p-1)$ and that contains $\{H, t_1\}$ as an invariant subgroup.

If $\{H, s_1^m t_1\}$ is contained as an invariant subgroup in a group that corresponds to P_{i_2}, then the former group must include the substitution

$$t_2^{-1} s_1^{-\beta} (s_1^m t_1) s_1^\beta t_2 = s_1^{m-\beta} s_r^\beta t_1^\gamma,$$

where β satisfies the congruence (2) and r denotes one of the numbers $2, 3 \ldots .,$ $p-1$. The group $\{H, s_1^m t_1\}$ contains the substitution $s_1^{m-\beta} s_r^\beta t_1$. If it also contained $s_1^{m-\beta} s_r^\beta t_1^\gamma$, then the substitution $t_1^{\gamma-1}$ would be found in it. This, however, is not the case since γ is not congruent to unity modulus p.

13. THEOREM.—*When $p > 2$, there is just one imprimitive group of degree pq that contains the head.*

$$H \equiv \{G_1, G_2, \ldots . , G_p\}_{\frac{q(q-1)}{2}, \frac{q(q-1)}{2}, \ldots , \frac{q(q-1)}{2}}$$

and whose substitutions interchange the systems of intransitivity of H according to P_{i_2}. *When* $p = 2$, *there are two such groups.*

The largest group within which the given head is invariant without having its systems interchanged is $\{G_1, G_2, \ldots, G_p\}$. This is of order $[q(q-1)]^p$, while H is of order $\left[\frac{q(q-1)}{2}\right]^p . 2$. There are accordingly 2^{p-1} sets of substitutions that permute according to each substitution in P_{i_2}. Those that transform according to p_1 may be obtained by multiplying the head by the substitutions

$$s_1^{\alpha_1} s_2^{\alpha_2} \ldots . s_{p-1}^{\alpha_{p-1}} t_1 \qquad \text{(A)}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p = 0$ or 1. Each of these transforms the head into itself, and each has its p^{th} power in the head. It may be noted, first, that all of these substitutions whose p^{th} power gives rise to the same substitution in the head generate, with H, conjugate groups. For, let $\alpha_1', \alpha_2', \ldots, \alpha_{p-1}'$ and $\alpha_1'', \alpha_2'', \ldots, \alpha_{p-1}''$ be two sets of values that satisfy the congruence

$$\alpha_1 + \alpha_2 + \ldots . + \alpha_{p-1} \equiv \alpha \qquad (\text{mod } q-1),$$

α being the exponent of the p^{th} power of the corresponding substitutions. The inverse of $s_2^{\beta_2} s_3^{\beta_3} \ldots . s_p^{\beta_p}$ is a substitution that transforms $\{H, s_1^{\alpha_1'} s_2^{\alpha_2'} \ldots . s_{p-1}^{\alpha_{p-1}'} t_1\}$ into $\{H, s_1^{\alpha_1''}, s_2^{\alpha_2''} \ldots . s_{p-1}^{\alpha_{p-1}''} t_1\}$ providing

$$\beta_i = (\alpha_1' + \alpha_2' + \ldots . + \alpha_{i-1}') - (\alpha_1'' + \alpha_2'' + \ldots . + \alpha_{i-1}'').$$

Of the substitutions (A) we need consider then only the following:

$$t_1 \text{ and } s_1 s_2 \ldots . s_x t_1,$$

where $x = 1, 2, \ldots, p-1$. It follows, by the method used above (§10), that the $p-1$ groups $\{H, s_1 s_2 \ldots . s_x t_1\}$ where $x = 1, 2, \ldots, p-1$, are conjugate. We have yet to consider the groups $\{H, t_1\}$ and $\{H, s_1 t_1\}$. The head H contains the substitution $s_1^{2\gamma_1+1} s_2^{2\gamma_2+1} \ldots . s_p^{2\gamma_p+1}$ where $\gamma_1, \gamma_2, \ldots, \gamma_p$ may be zero or any integer. Hence $\{H, s_1 t_1\}$ contains the substitution, $s_1^{2\gamma_1+2} s_2^{2\gamma_2+1} s_3^{2\gamma_3+1} \ldots . s_p^{2\gamma_p+1_2} t_1$. The p^{th} power of this will be identity if the γ's satisfy the congruence

$$2(\gamma_1 + \gamma_2 + \ldots . + \gamma_p) + p + 1 \equiv 0 \qquad (\text{mod } q-1)$$

i. e,

$$\gamma_1 + \gamma_2 \ldots . + \gamma_p = -\frac{p+1}{2} \qquad \left(\text{mod } \frac{q-1}{2}\right).$$

This evidently has a solution except where $p = 2$. Hence, where $p > 2$, the group $\{H, s_1 t_1\}$ contains a substitution of order p in the division in which $s_1 t_1$ occurs. In this case, the groups $\{H, t_1\}$ and $\{H, s_1 t_1\}$ are clearly conjugate. When $p = 2$, the group $\{H, s_1 t_1\}$ contains negative substitutions, while $\{H, t_1\}$ does not. Hence, when p is even, there are two groups with the given head that correspond to P_{p-1}, and when p is odd there is one such group.

The sets of substitutions that transform according to p_2 may be found by multiplying the head by the substitutions

$$s_2^{\alpha_2} s_3^{\alpha_3} \cdots\cdots s_p^{\alpha_p}\, t_2, \tag{B}$$

where $\alpha_2, \alpha_3, \ldots\ldots, \alpha_p = 0$ or 1. These all transform the head into itself. Now

$$t_2^{-1} s_2^{-\alpha_2} s_3^{-\alpha_3} \cdots\cdots s_p^{-\alpha_p}\, t_1\, s_2^{\alpha_2} s_3^{\alpha_3} \cdots\cdots s_p^{\alpha_p}\, t_2 = s_1^{\alpha_2}\, s_{i_2}^{\alpha_3 - \alpha_2} s_{i_3}^{\alpha_4 - \alpha_3} \cdots\cdots s_{i_{p-1}}^{\alpha_p - \alpha_{p-1}}\, s_{i_p}^{\alpha_p}\, t_1^{\gamma} \tag{C}$$

where $i_2, i_3, \ldots\ldots i_p = 2, 3, \ldots\ldots, p$ in some order and $t_2^{-1}\, t_1\, t_2 = t_1^{\gamma}$. Since $\alpha_2, \ldots\ldots, \alpha_p = 0$ or 1, it follows that the exponents of the s's that precede t_1^{γ} are either 0, 1, or -1. Further if (C) is found in $\{H, t_1\}$ when the exponent of one of the s's is zero they must all be zero. It follows that $\alpha_2 = \alpha_3 = \ldots\ldots = \alpha_p = 0$ is the only set of values which give a substitution (B) that transforms $\{H, t_1\}$ into itself. Hence there is just one group that contains the given head and corresponds to $P_{i_{i_2}}$ when i_{i_2} is less than $p - 1$.

14. THEOREM.—*When $p > 2$ and $\frac{p-1}{i_2}$ is even there are two imprimitive groups that contain the head*

$$H \equiv \{G_1, G_2, \ldots\ldots, G_p\}\ \text{pos},$$

and whose substitutions interchange the systems of intransitivity of H according to P_{i_2}; when $\frac{p-1}{i_2}$ is odd there is just one such group. When $p = 2$ there are two groups that contain the given head.

15. The heads considered in paragraphs 8–14 occur for all values of q and hence the theorems proved enable us to determine certain imprimitive groups of every degree of the form pq. In general, other intransitive groups can be formed from the p transitive groups $G_{1, i_1}, G_{2, i_1}, \ldots\ldots, G_{p, i_1}$ which may be used as heads of imprimitive groups whose systems of imprimitivity are permuted according to P_{i_2}. For each such head a like theorem may be proved.

SECTION IV. *List of the imprimitive groups of degree fifteen.*

The theorems proved in the preceding section enable us to find at once most of the imprimitive groups of the degrees four, six, nine, ten, and fourteen. We shall now make use of them in determining the imprimitive groups of degree fifteen.

Fifteen letters can be divided in two ways into systems containing an equal number of letters, viz., into three systems of five letters each, or into five systems of three letters each. We consider first the groups that contain three systems of imprimitivity. The substitutions of these groups permute the systems according to either the symmetric group or the alternating group of degree three. It follows that any intransitive group of degree fifteen that can be used as the head of such a group must permit a cyclic interchange of its systems of intransitivity. It is not difficult to construct all the intransitive groups of degree fifteen having three systems of intransitivity that have this property. It is found that there are twenty-one such groups which can be used as heads. They are as follows:

Order.	
1728000	$(abcde)$ all $(fghij)$ all $(klmno)$ all
864000	$\{(abcde)$ all $(fghij)$ all $(klmno)$ all$\}$ pos
432000	$(abcde)$ pos $(fghij)$ pos $(klmno)$ pos +
	$(abcde)$ neg $(fghij)$ neg $(klmno)$ neg
216000	$(abcde)$ pos $(fghij)$ pos $(klmno)$ pos
8000	$(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}$
4000	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}$ pos
2000	$[\{(abcde)_{20}\ (fghij)_{20}\}$ pos, $(klmno)_{20}]$ dim
2000	$\{(abcde)_{20}\ (fghij)_{20},\ (klmno)_{20}\}_{100,\,5}$
1000	$(abcde)_{10}\ (fghij)_{10}\ (klmno)_{10}$
500	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}_{5,\,5,\,5}$
500	$\{(abcde)_{10}\ (fghij)_{10},\ (klmno)_{10}\}$ dim
250	$\{(abcde)_{10}\ (fghij)_{10}\ (klmno)_{10}\}_{5,\,5,\,5}$
125	$(abcde)_5\ (fghij)_5\ (klmno)_5$
120	$(abcde\,.\,fghij\,.\,klmno)_{120}$
100	$[\{(abcde)_{20}\ (fghij)_{20}\}_{5,\,5},\ (klmno)_{20}]_{5,\,1}$
60	$(abcde\,.\,fghij\,.\,klmno)_{60}$
50	$[\{(abcde)_{10}\ (fghij)_{10}\}$ dim, $(klmno)_{10}]_{5,\,1}$

25 $\{(abcde)_5\ (fghij)_5,\ (klmno)\}_{5,1}$
20 $(abcde\,.\,fghij\,.\,klmno)_{20}$
10 $(abcde\,.\,fghij\,.\,klmno)_{10}$
5 $(abcde\,.\,fghij\,.\,klmno)_5$

The theorems of the preceding section enable us to write down at once all of the imprimitive groups that have the above groups for heads except the second head of order 500. It is easily found that there are three groups that contain this head. The total number of these is found to be 55 and they may be written as follows:

Order.	No.	
10368000	1	$(abcde)$ all $(fghij)$ all $(klmno)$ all $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)$
5184000	1	$(abcde)$ all $(fghij)$ all $(klmno)$ all $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
	2, 3	$\{(abcde)$ all $(fghij)$ all $(klmno)$ all$\}$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)(1,\ ab)$
2592000	1	$\{(abcde)$ all $(fghij)$ all $(klmno)$ all$\}$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
2592000	2	$(abcde)$ pos $(fghij)$ pos $(klmno)$ pos $+$ $(abcde)$ neg $(fghij)$ neg $(klmno)$ neg $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)$
1296000	1	$\{(abcde)$ pos $(fghij)$ pos $(klmno)$ pos $+$ $(abcde)$ neg $(fghij)$ neg $(klmno)$ neg$\}$ $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
	2, 3	$(abcde)$ pos $(fghij)$ pos $(klmno)$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)(1,\ ab\,.\,fg\,.\,kl)$
648000	1	$(abcde)$ pos $(fghij)$ pos $(klmno)$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
48000	1	$(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)$
24000	1	$(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
	2	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)$
	3	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(bced)(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)$
12000	1	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}$ pos $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
	2	$[\{(abcde)_{20}\ (fghij)_{20}\}$ pos, $(klmno)_{20}]$ dim $(afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)$

12000	3, 4	$\{(abcde)_{20}\ (fghij)_{20},\ (klmno)_{20}\}_{100,\,5}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ lo.mn)$
6000	1	$[\{(abcde)_{20}\ (fghij)_{20}\}\ \mathrm{pos},\ (klmno_{20})]\ \mathrm{dim}\ (afk.bgl.chm.din.ejo)$
	2	$\{(abcde)_{20}\ (fghij)^{20},\ (klmno)_{20}\}_{100,\,5}\ (afk.bgl.chm.din.ejo)$
	3, 4	$(abcde)_{10}\ (fghij)_{10}\ (klmno)_{10}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ bced.ghji.lmon)$
3000	1	$(abcde)_{10}\ (fghij)_{10}\ (klmno)_{10}\ (afk.bgl.chm.din.ejo)$
	2	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}_{5,\,5,\,5}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)$
	3, 4	$\{(abcde)_{10}\ (fghij)_{10},\ (klmno)_{10}\}\ \mathrm{dim}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ be.cd)$
1500	1	$\{(abcde)_{20}\ (fghij)_{20}\ (klmno)_{20}\}_{5,\,5,\,5}\ (afk.bgl.chm.din.ejo)$
	2	$\{(abcde)_{10}\ (fghij)_{10},\ (klmno)_{10}\}\ \mathrm{dim}\ (afk.bgl.chm.din.ejo)$
	3, 4	$\{(abcde)_{10}\ (fghij)_{10}\ (klmno)_{10}\}_{5,\,5,\,5}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ bced.ghji.lmon)$
750	1	$\{(abcde)_{10}\ (fghij)_{10}\ (klmno)_{10}\}_{5,\,5,\,5}\ (afk.bgl.chm.din.ejo)$
	2, 3	$(abcde)_{5}\ (fghij)_{5}\ (klmno)_{5}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ be.cd.gj.hi.lo.mn)$
720	1	*$(abcde.fghij.klmno)_{120}\ (afk.bgl.chm.din.ejo)(af.bg.ch.di.ej)$
600	1	$[\{(abcde)_{20}\ (fghij)_{20}\}_{5,\,5},\ (klmno)_{20}]_{5,\,1}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)$
375	1	$(abcde)_{5}\ (fghij)_{5}\ (klmno)_{5}\ (afk.bgl.chm.din.ejo)$
360	1	*$(abcde.fghij.klmno)_{120}\ (afk.bgl.chm.din.ejo)$
	2, 3	*$(abcde.fghij.klmno)_{60}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ ab.fg.kl)$
300	1	$[\{(abcde)_{20}\ (fghij)_{20}\}_{5,\,5}\ (klmno)_{20}]_{5,\,1}\ (afk.bgl.chm.din.ejo)$
	2, 3	$[\{(abcde)_{10}\ (fghij)_{10}\}\ \mathrm{dim},\ (klmno)_{10}]_{5,\,1}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ bced.ghji.lmon)$
180	1	*$(abcde.fghij.klmno)_{60}\ (afk.bgl.chm.din.ejo)$
150	1	$[\{(abcde)_{10}\ (fghij)_{10}\}\ \mathrm{dim},\ (klmno)_{10}]_{5,\,1}\ (afk.bgl.chm.din.ejo)$
	2, 3	$\{(abcde)_{5}\ (fohij)_{5},\ (klmno)_{5}\}_{5,\,1}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)(1,\ be.cd.gj.hi.lo.mn)$
120	1	*$(abcde.fghij.klmno)_{20}\ (afk.bgl.chm.din.ejo)$
		$(af.bg.ch.di.ej)$

75	1	$\{(abcde)_5\ (fghij)_5,\ (klmno)_5\}_{5,\,1}\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
60	1	$*(abcde\,.\,fghij\,.\,klmno)_{20}\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
	2, 3	$*(abcde\,.\,fghij\,.\,klmno)_{10}\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)(1,\ bced\,.\,ghji\,.\,lmon)$
30	1	$*(abcde\,.\,fghij\,.\,klmno)_{10}\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
	2, 3	$*(abcde\,.\,fghij\,.\,klmno)_5\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$ $(af\,.\,bg\,.\,ch\,.\,di\,.\,ej)(1,\ be\,.\,cd\,.\,gj\,.\,hi\,.\,lo\,.\,mn)$
15	1	$*(abcde\,.\,fghij\,.\,klmno)_5\ (afk\,.\,bgl\,.\,chm\,.\,din\,.\,ejo)$
Total,	55	

Those groups marked with an * have also five systems of imprimitivity.

The symmetric group and the alternating group of degree five can each be represented as an imprimitive group of degree fifteen. Hence identity occurs among the heads of the imprimitive groups of degree fifteen that have five systems of imprimitivity. The transitive constituents of the other heads of these groups are cyclic groups of order three or symmetric groups of order six. And as in the preceding case each of these heads permits a cyclic interchange of its systems of intransitivity. It is found that there are nine groups that can be used for heads of imprimitive groups of degree fifteen that have five systems of imprimitivity. They are as follows:

Order.	
7776	(abc) all (def) all (ghi) all (jkl) all (mno) all
3888	$\{(abc)$ all (def) all (ghi) all (jkl) all (mno) all$\}$ pos
486	$\{(abc)$ all (def) all (ghi) all (jkl) all (mno) all$\}_{3,\,3,\,3,\,3,\,3}$
243	$(abc)(def)(ghi)(jkl)(mno)$
162	$(abc\,.\,dfe)(def\,.\,gih)(ghi\,.\,jlk)(jkl\,.\,mon)(ab\,.\,de\,.\,gh\,.\,jk\,.\,mn)$
81	$(abc\,.\,dfe)(def\,.\,gih)(ghi\,.\,jlk)(jkl\,.\,mon)$
6	$(abc\,.\,def\,.\,ghi\,.\,jkl\,.\,mno)$ all
3	$(abc\,.\,def\,.\,ghi\,.\,jkl\,.\,mno)$ cyc
1	Identity

The theorems of the preceding section enable us to determine all the imprimitive groups that contain these heads except those whose substitutions interchange their systems of imprimitivity according to either the symmetric or the

alternating group of degree five. We note briefly the construction of the latter. Those that contain the head identity can be found at once by means of Dyck's theorem on the transitive representation of a given group.* The heads of orders 7776 and 6 are not contained in larger groups of the same degree that leave their systems of intransitivity unchanged. The groups that contain these heads and that correspond to $(abcde)$ all or $(abcde)$ pos are determined then at once.† The groups that contain the head of order 3 and that correspond to $(abcde)$ all or $(abcde)$ pos are of order 360 or 180 and their factors of composition are 60, 3, 2 and 60, 3 respectively. The abstract groups with these factors of composition are known,‡ and hence we can at once find the corresponding imprimitive groups. The remaining groups (12 in number) may be easily found by tentative processes.

I find in all 56 distinct groups that contain five systems of imprimitivity. Of these, 13 have also three systems of imprimitivity and so are found in the preceding list. Those which contain five systems of imprimitivity without also containing three systems are the following:

Order	No.	
933120	1	(abc) all (def) all (ghi) all (jkl) all (mno) all $(adgjm \,.\, behkn \,.\, cfilo)(ad \,.\, be \,.\, cf)$
466560	1	(abc) all (def) all (ghi) all (jkl) all (mno) all $(adgjm \,.\, behkn \,.\, cfilo)(adg \,.\, beh \,.\, cfi)$
	2, 3	$\{(abc)$ all (def) all (ghi) all (jkl) all (mno) all$\}$ pos $(adgjm \,.\, behkn \,.\, cfilo)(ad \,.\, be \,.\, cf)(1, gh)$
233280	1	$\{(abc)$ all (def) all (ghi) all (jkl) all (mno) all$\}$ pos $(adgjm \,.\, behkn \,.\, cfilo)(adg \,.\, beh \,.\, cfi)$
155520	1	(abc) all (def) all (ghi) all (jkl) all (mno) all $(adgjm \,.\, behkn \,.\, cfilo)(dgmj \,.\, ehnk \,.\, fiol)$
77760	1	(abc) all (def) all (ghi) all (jkl) all (mno) all $(adgjm \,.\, behkn \,.\, cfilo)(dm \,.\, gj \,.\, en \,.\, hk \,.\, fo \,.\, il)$
	2, 3	$\{(abc)$ all (def) all (ghi) all (jkl) all (mno) all$\}$ pos $(adgjm \,.\, behkn \,.\, cfilo)(dgmj \,.\, ehnk \,.\, fiol)(1, ab)$

* Mathematische Annalen, Vol. 22 (1883), p. 94.

† Miller, Quarterly Journal of Mathematics, Vol. 28 (1895), p. 195.

‡ Hölder, Mathematische Annalen, Vol. 46 (1895), p. 417.

58320	1	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all}$_{3,3,3,3,3}$ (*adgjm* . *behkn* . *cfilo*)(*ad* . *be* . *cf*)
38880	1	(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all (*adgjm* . *behkn* . *cfilo*)
	2, 3	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all} pos (*adgjm* . *behkn* . *cfilo*)(*dm* . *gj* . *en* . *hk* . *fo* . *il*)(1, *ab*)
29160	1	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all}$_{3,3,3,3,3}$ (*adgjm* . *behkn* . *cfilo*)(*adg* . *beh* . *cfi*)
	2, 3	(*abc*)(*def*)(*ghi*)(*jkl*)(*mno*)(*adgjm* . *behkn* . *cfilo*) (*ad* . *be* . *cf*)(1, *ab* . *de* . *gh* . *jk* . *mn*)
19440	1	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all} pos (*adgjm* . *behkn* . *cfilo*)
	2	(*abc* . *dfe*)(*def* . *gih*)(*ghi* . *jlk*)(*jkl* . *mon*)(*ab* . *de* . *gh* . *jk* . *mn*) (*adgjm* . *behkn* . *cfilo*)(*ad* . *be* . *cf*)
14580	1	(*abc*)(*def*)(*ghi*)(*jkl*)(*mno*)(*adgjm* . *behkn* : *cfilo*)(*adg* . *beh* . *cfi*)
9720	1	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all}$_{3,3,3,3,3}$ (*adgjm* . *behkn* . *cfilo*)(*dgmj* . *ehnk* . *fiol*)
	2	(*abc* . *dfe*)(*def* . *gih*)(*ghi* . *jlk*)(*jkl* . *mon*)(*ab* . *de* . *gh* . *jk* . *mn*) (*adgjm* . *behkn* . *cfilo*)(*adg* . *beh* . *cfi*)
	3, 4	(*abc* . *dfe*)(*def* . *gih*)(*ghi* . *jlk*)(*jkl* . *mon*)(*adgjm* . *behkn* . *cfilo*) (*ad* . *be* . *cf*)(1, *ab* . *de* . *gh* . *jk* . *mn*)
4860	1	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all}$_{3,3,3,3,3}$ (*adgjm* . *behkn* . *cfilo*)(*dm* . *gj* . *en* . *hk* . *fo* . *il*)
	2, 3	(*abc*)(*def*)(*ghi*)(*jkl*)(*mno*)(*adgjm* . *behkn* . *cfilo*) (*dgmj* . *ehnk* . *fiol*)(1, *ab* . *de* . *gh* . *jk* . *mn*)
	4	(*abc* . *dfe*)(*def* . *gih*)(*ghi* . *jlk*)(*jkl* . *mon*)(*adgjm* . *behkn* . *cfilo*) (*adg* . *beh* . *cfi*)
3240	1	(*abc* . *dfe*)(*def* . *gih*)(*ghi* . *jlk*)(*jkl* . *mon*)(*ab* . *de* . *gh* . *jk* . *mn*) (*adgjm* . *behkn* . *cfilo*)(*dgmj* . *ehnk* . *fiol*)
2430	1	{(*abc*) all (*def*) all (*ghi*) all (*jkl*) all (*mno*) all}$_{3,3,3,3,3}$ (*adgjm* . *behkn* . *cfilo*)
	2, 3	(*abc*)(*def*)(*ghi*)(*jkl*)(*mno*)(*adgjm* . *behkn* . *cfilo*) (*dm* . *gj* . *en* . *hk* . *fo* . *il*)(1, *ab* . *de* . *gh* . *jk* . *mn*)
1620	1	(*abc* . *dfe*)(*def* . *gih*)(*ghi* . *jlk*)(*jkl* . *mon*)(*ab* . *de* . *gh* . *jk* . *mn*) (*adgjm* . *behkn* . *cfilo*)(*dm* . *gj* . *en* . *hk* . *fo* . *il*)

1620,	3	$(abc \,.\, dfe)(def \,.\, gih)(ghi \,.\, jlk)(jkl \,.\, mon)(adgjm \,.\, behkn \,.\, cfilo)$ $(dgmj \,.\, ehnk \,.\, fiol)(1, ab \,.\, de \,.\, gh \,.\, jk \,.\, mn)$
1215	1	$(abc)(def)(ghi)(jkl)(mno)(adgjm \,.\, behkn \,.\, cfilo)$
810	1	$(abc \,.\, dfe)(def \,.\, gih)(ghi \,.\, jlk)(jkl \,.\, mon)(ab \,.\, de \,.\, gh \,.\, jk \,.\, mn)$ $(adgjm \,.\, behkn \,.\, cfilo)$
	2, 3	$(abc \,.\, dfe)(def \,.\, gih)(ghi \,.\, jlk)(jkl \,.\, mon)(adgjm \,.\, behkn \,.\, cfilo)$ $(dm \,.\, gj \,.\, en \,.\, hk \,.\, fo \,.\, il)(1, ab \,.\, de \,.\, gh \,.\, jk \,.\, mn)$
405	1	$(abc \,.\, dfe)(def \,.\, gih)(ghi \,.\, jlk)(jkl \,.\, mon)(adgjm \,.\, behkn \,.\, cfilo)$
360	1	$(abc \,.\, def \,.\, ghi \,.\, jkl \,.\, mno)$ cyc $(adgjm \,.\, behkn \,.\, cfilo)$ $(def \,.\, gkn \,.\, hlo \,,\, ijm)$
180	1	$(abc \,.\, def \,.\, ghi \,.\, jkl \,.\, mno)$ cyc $(adgjm \,.\, behkn \,.\, cfilo)$ $(aeh \,.\, bfi \,.\, cdg \,.\, mno)$
120	1	$(adgjm \,.\, behkn \,.\, cfilo)(ad \,.\, be \,.\, cf)$
60	1	$(adgjm \,.\, behkn \,.\, cfilo)(adg \,.\, beh \,.\, cfi)$
	43	

CORNELL UNIVERSITY, *June*, 1901.

On Nullsystems in Space of Five Dimensions and their Relation to Ordinary Space.

By John Eiesland.

The investigations contained in the following pages are chiefly concerned with certain nullsystems in hyperspace. The reason for the special interest in the case $n = 5$ is obvious from the fact that the geometry of point-manifoldnesses in such a space becomes by means of Lie's transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_5 + x_1x_2 + x_3x_4 = X_3,$$

a geometry of surface-elements in ordinary space just as the geometry of ordinary space by the analogous transformation for $n = 3$ employed by Lie in his "Geometrie der Berührungstransformationen" becomes the geometry of line-elements in the plane.

Closely associated with a reduced nullsystem

$$x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 + dx_5 = 0,$$

is a complex whose lines are lines of this system and at the same time satisfy the Monge equation

$$dx_1dx_2 + dx_3dx_4 = 0.$$

I have called this an *asymptotic complex* owing to the close relation which exists between it and the linear tangents along asymptotic lines on a surface in ordinary space.

The paper has been divided into three parts. In the first, the general nullsystem in n-dimensional space (n odd) has been derived by means of a Euclidian motion and reduced to its simplest form. Part II discusses the transformation of lines of the nullsystem in five-dimensional space into certain configurations of surface-elements in ordinary space. The problem to find all the two-dimensional surfaces in M_5 whose coordinate lines (u) and (v) belong to an asymptotic

complex, is next taken up, and it is shown that its solution depends on the integration of a differential equation of the second order with equal invariants. Certain applications of asymptotic complexes to surface theory are also given. Finally, in the third part, the question of invariance of the nullsystem and the asymptotic complex, when subjected to projective, and in particular to Euclidian transformations, has been treated from the standpoint of Lie's group-theory. Certain theorems concerning contact-transformations in ordinary space and theorems concerning the mobility of a nullsystem have been obtained.

I. We shall define an infinitesimal motion in the space M_n as an infinitesimal point-transformation that does not alter the distance between two consecutive points. Let there be given the following system of equations:

$$\delta x_i = \xi_i (x_1, x_2, \ldots\ldots, x_n)\, \delta t, \qquad (i = 1, 2, \ldots\ldots, n), \tag{1}$$

defining an infinitesimal transformation in time δt. In order that the distance shall remain invariant, these equations must satisfy the following relations:

$$\delta (dx_1^2 + dx_2^2 + \ldots\ldots dx_n^2) \equiv 0,$$

or, what is the same thing,

$$dx_1\, \delta dx_1 + dx_2\, \delta dx_2 — \ldots\ldots dx_n\, \delta dx_n = 0.$$

Now since
$$dx_i\, \delta dx_i = dx_i\, d\delta x_i,$$

we get the following conditions:

$$dx_1\, \delta\xi_1 + dx_2\, \delta\xi_2 + \ldots\ldots dx_n\, \delta x_n = 0, \tag{2}$$

but we have also

$$\delta\xi_i = \frac{\partial\xi_i}{\partial x_1} dx_1 + \frac{\partial\xi_i}{\partial x_2} dx_2 + \ldots\ldots + \frac{\partial\xi_i}{\partial x_n} dx_n \, (i = 1, 2, \ldots\ldots, n)$$

and on substituting these values of $\delta\xi_i$ in (2) and equating to zero the coefficients of dx_i^2 and $dx_i dx_k$, we obtain the following relations:

$$\frac{\partial\xi_1}{\partial x_1} + \frac{\partial\xi_2}{\partial x_2} = \ldots\ldots = \frac{\partial\xi_n}{\partial x_n} = 0, \tag{3}$$

$$\frac{\partial\xi_i}{\partial x_k} + \frac{\partial\xi_k}{\partial x_i} = 0, \quad i \not< k; \tag{4}$$

from (3) it follows that the functions $\xi_1, \xi_2 \dots \xi_n$ do not contain the variables $x_1, x_2 \dots x_n$ respectively. Differentiating the equations (4) partially with respect to x_k we obtain

$$\frac{\partial}{\partial x_k}\frac{\partial \xi_i}{\partial x_k} + \frac{\partial}{\partial x_k}\frac{\partial \xi_k}{\partial x_i} = 0 .$$

Now since the second term on the left hand side is equal to zero, we have

$$\frac{\partial^2 \xi_i}{\partial x_k^2} = 0$$

which shows that the ξ_i's are linear in the x's. The equations (1) therefore reduce to the following

$$\left.\begin{array}{ll} \dfrac{\delta x_1}{\delta t} = & p_{12}x_2 + p_{13}x_3 + \dots + p_{1n}x_n \qquad + c_1 , \\ \dfrac{\delta x_2}{\delta t} = & -p_{12}x_1 + p_{23}x_3 + \dots + p_{2n}x_n \qquad + c_2 , \\ \dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\delta x_n}{\delta t} = & -p_{1n}x_1 - p_{2n}x_2 - \dots - p_{n-1}\ x_{n-1} + c_n , \end{array}\right\} \tag{5}$$

in which p_{ik}, are arbitrary constants.

From the form of the above system we observe that *the most general infinitesimal motion in space of n dimensions having a distance-invariant*

$$= \sqrt{dx_1^2 + dx^2 + \dots + dx_n^2} = const.$$

is made up of a rotation about the origin transforming the hypersphere

$$x_1^2 + x_2^2 + \dots + x_n^2 = const.$$

into itself, and n translations along the n axes respectively. In fact, if we let

$$c_1 = c_2 = \dots = c_n = 0$$

and then multiply the first equation by x_1, the second by x_2 and so on, we obtain

$$x_1\frac{\delta x_1}{\delta t} + x_2\frac{\delta x_2}{\delta t} + \dots + x_n\frac{\delta x_n}{\delta t} = 0$$

or,
$$x_1^2 + x_2^2 + \dots + x_n^2 = \text{const.}$$ Q. E. D.

We shall now consider the ∞^n linear spaces

$$D_1x_1 + D_2x_2 + \dots + D_nx_n = C, \tag{6}$$

whose n direction-cosines are proportional to D_1, $D_2 \ldots . D_n$ respectively. Among all these spaces does there exist one which is translated parallel to itself when subjected to the infinitesimal motion (5)? If so it is clear that the following condition must be satisfied

$$D_1 \frac{\delta x_1}{\delta t} + D_2 \frac{\delta x_2}{\delta t} + \ldots . + D_n \frac{\delta x_n}{\delta t} = 0,$$

which, by the use of the equations (5), give rise to the following set of equations:

$$\left.\begin{array}{l} D_2 p_{12} + D_3 p_{13} + \ldots . + D_n p_{1n} \qquad = 0, \\ -D_1 p_{12} + D_3 p_{25} + \ldots . + D_n p_{2n} \qquad = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -D_1 p_{1n} - D_2 p_{2n} - \ldots . - D_{n-1} p_{n-1n} = 0. \end{array}\right\} \tag{7}$$

We shall now consider two cases: n even and n odd. 1°. n even. In this case the determinant of the above system, being a zero-axial skew-determinant of even order, is equal to the square of a certain integral function of the elements (the so-called Pfaffian function), and since the p's are supposed to be all independent of each other, the determinant does not vanish. Hence, there exist in general no values of the D's which will satisfy the above equations, so that *in space of even dimensions no parallel motion of a linear space is possible in general.*

Remark. This statement ceases to be true if the determinant of the p's vanish; but in this case the motion is no longer general; thus for $n = 2$ we have a translation along the axes.

2°. n odd. In this case the skew-determinant vanishes identically; the ratios $\frac{D_1}{D_n}, \frac{D_2}{D_n}, \ldots, \frac{D_{n-1}}{D_n}$ may therefore be found and the problem is possible, provided, of course, not all the minors of order $n-1$ vanish. Having determined the D's to within a factor of proportionality by solving the system (7), we shall next consider all the points in the space M_n moving perpendicular to the space (6). For all such points $\delta x_1, \delta x_2, \ldots, \delta x_n$ must be proportional to the direction-cosines of the linear space; that is, we must have

$$\left.\begin{array}{ll} p_{12}x_2 + p_{13}x_3 + \cdots\cdots + p_{1n}x_n + c_1 & = \rho D_1, \\ -p_{12}x_1 + p_{23}x_3 + \cdots\cdots + p_{2n}x_n + c_2 & = \rho D_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -p_{1n}x_1 - p_{2n}x_2 - \cdots\cdots - p_{n-1\,n}x_{n-1} + c_n & = \rho D_n. \end{array}\right\} \tag{8}$$

Multiplying these equations by $D_1, D_2, \ldots\ldots, D_n$ respectively and adding, we obtain

$$c_1D_1 + c_2D_2 + \cdots\cdots + c_nD_n = \rho\,[D_1^2 + D_2^2 + \cdots\cdots + D_n^2],$$

which equality determines the factor of proportionality ρ, provided we have

$$D_1^2 + D_2^2 + \cdots\cdots + D_n^2 \neq 0.$$

The system (8) is thus seen to contain only $n-1$ independent equations and n unknowns, and represents, therefore, a one-dimensional manifoldness or a straight line. This line will, by the infinitesimal motion (5), be transformed into itself and is, therefore, *the invariant line.**

Let us now choose this invariant line as our x_n-axis, and let all the other axes be perpendicular to it. Since the x_n-axis is invariant, we must have $\delta x_1 = \delta x_2 = \ldots\ldots = \delta x_{n-1} = 0$ for $x_1 = x_2 = \ldots\ldots = x_{n-1} = 0$, that is,

$$p_{1n} = p_{2n} = p_{3n} = \cdots\cdots = p_{n\,n-1} = c_1 = c_2 = \cdots\cdots = c_{n-1} = 0$$

and our system takes the form

$$\left.\begin{array}{ll} \dfrac{\delta x_1}{\delta t} = & p_{12}x_2 + p_{13}x_3 + \cdots\cdots + p_{1\,n-1}x_{n-1}, \\ \dfrac{\delta x_2}{\delta t} = & -p_{12}x_1 + p_{23}x_3 + \cdots\cdots + p_{2\,n-1}x_{n-1}, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\delta x_n}{\delta t} = & c_n. \end{array}\right\} \tag{9}$$

---

* If $\sum\limits_1^n D_n^2 = 0$, the space $\sum\limits_1^n {}_n x_n =$ const. will be tangent to the element-hypercone $dx_1^2 + dx_2^2 + \ldots\ldots + dx_n^2 = 0$. In fact, in order that the equations $\sum D_n dx_n = 0$ and $\sum dx_n^2 = 0$ shall have a common solution, we must have $dx_1 : dx_2 : \ldots\ldots : dx_n = D_1 : D_2 : \ldots\ldots : D_n$; that is, $D_1^2 + D_2^2 + \ldots\ldots + D_n^2 = 0$. This is therefore the condition for minimal lines; in particular, the invariant line (8) can be no such line.

Of these equations the first $n-1$ satisfy the relation

$$x_1\delta x_1 + x_2\delta x_2 + \ldots\ldots + x_{n-1}\delta x_{n-1} = 0,$$

or

$$x_1^2 + x_2^2 + \ldots\ldots + x_{n-1}^2 = \text{const};$$

hence we obtain the following

THEOREM.—*The most general infinitesimal motion in M_n (n odd) may be decomposed into two distinct motions: a motion leaving the origin at rest and a translation parallel to the invariant axis.*

The above system (9) can be simplified still further. Starting with the original system we shall apply a purely analytical method using the following orthogonal transformation of coordinates

$$\left.\begin{array}{l} x_1 = a_{11}x_1' + a_{12}x_2' + \ldots\ldots + a_{1n}x_n' + k_1, \\ x_2 = a_{21}x_1' + a_{22}x_2' + \ldots\ldots + a_{2n}x_n' + k_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = a_{n1}x_1' + a_{n2}x_2' + \ldots\ldots + a_{nn}x_n' + k_n, \end{array}\right\} \quad (10)$$

in which the determinant $|a_{11}, a_{22}, \ldots\ldots, a_{nn}| = 1$ and also

$$\left.\begin{array}{l} \sum\limits_{s=1}^{s=n} a_{1s}^2 = 1, \ \sum\limits_{s=1}^{s=n} a_{2s}^2 = 1, \ \ldots\ldots, \ \sum\limits_{s=1}^{s=n} a_{ns}^2 = 1 \\ \sum\limits_{t=1}^{t=n} a_{kt}a_{st} = 0, \quad \left[\begin{array}{l} k = 1, 2, \ldots\ldots, n \\ s = 1, 2, \ldots\ldots, n \\ k \neq s \end{array}\right]; \end{array}\right\} \quad (11)$$

we shall also put

$$P_{ik}^{st} = \begin{vmatrix} a_{is} & a_{it} \\ a_{ks} & a_{kt} \end{vmatrix}, \quad \left[\begin{array}{l} i, k = 1, 2 \ldots\ldots, n \\ s, t = 1, 2 \ldots\ldots, n \end{array}\right]$$

from which it follows that $P_{ik}^{st} = -P_{ik}^{ts}$, $P_{ii}^{st} = 0$ and $P_{ik}^{ss} = 0$.

The system (10) contains $n^2 + n$ parameters all of which are not independent since there exist $\frac{n(n-1)}{2} + n$ relations between them. There remain then $\frac{n(n+1)}{2}$ independent parameters which are functions of the n^2 direction-cosines and of the k's. Introducing now the new coordinates we have

$$\delta x_1' = \sum_1^n a_{s1}\, \delta x_s,$$

$$\delta x_2' = \sum_1^n a_{s2}\, \delta x_s,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\delta x_n' = \sum_1^n a_{sn}\, \delta x_s.$$

Substituting on the right side the values of δx_s, introducing at the same time the values of $x_1, x_2, \ldots, x$ obtained from (10), we obtain, after some easy reductions,

$$\left.\begin{aligned}
\frac{\delta x_1'}{\delta t} &= \sum_{i,k}^{1\ldots n} p_{ik} P_{ik}^{12} x_2' + \sum p_{ik} P_{ik}^{13} x_3' + \ldots + \sum p_{in} P_{ik}^{in} x_n' + C_1, \\
\frac{\delta x_2'}{\delta t} &= -\sum p_{ik} P_{ik}^{12} x_1' + \sum p_{ik} P_{ik}^{23} x_3' + \ldots + \sum p_{ik} P_{ik}^{2n} x_n' + C_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\delta x_n'}{\delta t} &= -\sum p_{ik} P_{ik}^{1n} x_1' + \sum p_{ik} P_{ik}^{2n} x_2' + \ldots - \sum p_{ik} P_{ik}^{n-1\,n} x_{n-1}' + C_n,
\end{aligned}\right\} \quad (12)$$

in which $i \neq k$. The C's may be expressed by the following system of equations:

$$C_1 = \sum_1^n a_{s1} \frac{\delta k_s}{\delta t},$$

$$C_2 = \sum a_{s2} \frac{\delta k}{\delta t},$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$C_n = \sum a_{sn} \frac{\delta k_s}{\delta t},$$

where $\frac{\delta k_s}{\delta t}$ stands for what is obtained by substituting for $x_1, x_2 \ldots x_n$ the

quantities $k_1, k_2 \ldots . k_n$ respectively in the original system for the motion. Solving, we find,

$$\frac{\delta k_1}{\delta t} = C_1 a_{11} + C_2 a_{12} + \ldots . + C_{1n} a_{1n},$$

$$\frac{\delta k_2}{\delta t} = C_1 a_{21} + C_2 a_{22} + \ldots . + C_{2n} a_{2n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\delta k_n}{\delta t} = C_1 a_{n1} + C_2 a_{n2} + \ldots . + C_n a_{nn}.$$

The C's may now be disposed of by putting all of them equal to zero except one, say C_n. We have then

$$\left.\begin{array}{ll} k_2 p_{12} + k_3 p_{13} + \ldots . + k_n p_{1n} + c_1 & = C_n a_{1n}, \\ -k_1 p_{12} + k_3 p_{23} + \ldots . + k_n p_{2n} + c_2 & = C_n a_{2n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ -k_1 p_{1n} - k_2 p_{2n} - \ldots . - k_{n-1} p_{n-1\,n} + c_n & = C_n a_{nn}, \end{array}\right\} \quad (13)$$

In order that this system may be satisfied for finite values of the k's, we must have

$$c_1 D_1 + c_2 D_2 + \ldots . c_n D_n = C_n [a_{1n} D_1 + a_{2n} D_2 + \ldots . + a_{nn} D_n],$$

where the D's are the minors of the elements of the first column in the abovementioned skew-symmetrical determinant of the p's. We may now dispose of the a's by putting

$$a_{1n} = \frac{D_1}{\sqrt{D_1^2 + D_2^2 + \ldots . + D_n^2}},$$

$$a_{2n} = \frac{D_2}{\sqrt{D_1^2 + D_2^2 + \ldots . + D_n^2}}, \ldots a_{nn} = \frac{D_n}{\sqrt{D_1^2 + D_2^2 + \ldots . + D_n^2}}, \quad (14)$$

and since by hypothesis $D_1^2 + D_2^2 + \ldots . + D_n^2 \neq 0$, we obtain the following value for C_n

$$C_n = \frac{c_1 D_1 + c_2 D_2 + \ldots . + c_n D_n}{\sqrt{D_1^2 + D_2^2 + \ldots . + D_n^2}}.$$

The system (13) shows that the point $(k_1, k_2, \ldots . , k_n)$ must be situated on a

straight line which is in fact nothing but the invariant axis obtained before, as is easily verified by comparing the system (13) with (8); we also observe that $C_n = \rho\sqrt{D_1^2 + D_2^2 + \dots + D_n^2}$.

If now we introduce the values of a_{1n}, $a_{2n} \dots a_{nn}$ given by equations (14) into the equations (12) and reduce by means of (11), putting also at the same time $\Sigma p_{ik} P_{ik}^{12} \delta t$ equal to a new δt, we obtain the following system

$$\left.\begin{aligned}
\frac{\delta x_1}{\delta t} &= \quad x_2 + \frac{\sum p_{ik} P_{ik}^{13}}{\sum p_{ik} P_{ik}^{12}} x_3 + \dots + \frac{\sum p_{ik} P_{ik}^{1n-1}}{\sum p_{ik} P_{ik}^{12}} x_{n-1}, \\
\frac{\delta x_2}{\delta t} &= -x_1 + \frac{\sum p_{ik} P_{ik}^{12}}{\sum p_{ik} P_{ik}^{12}} x_3 + \dots + \frac{\sum p_{ik} P_{ik}^{2n-1}}{\sum p_{ik} P_{k}^{12}} x_{n-1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{\delta x_n}{\delta t} &= C_n = \rho\sqrt{D_1^2 + D_2^2 + \dots + D_n^2},
\end{aligned}\right\} \qquad (12')$$

which is of the same form as the system (9). Since the direction-cosines $a_{1n}, a_{2n}, \dots, a_{nn}$ have already been determined, there remain $n^2 - n$ with $2(n-1) + \frac{(n-1)(n-2)}{2}$ relations between them, so that there are in the system $\frac{(n-1)(n-2)}{2}$ essential parameters,* or one more than the number of coefficients in the system (12'), not counting C_n. We are, therefore, able to dispose of these at will, provided we do not make the determinant of the system vanish. We may thus put

$$\frac{\sum p_{ik} P_{ik}^{37}}{\sum p_{ik} P_{ik}^{12}} = 1, \quad \frac{\sum p_{ik} P_{ik}^{36}}{\sum p_{ik} P_{ik}^{12}} = 1, \dots \frac{\sum p_{ik} P_{ik}^{n-2,\, n-1}}{\sum p_{ik} P_{ik}^{12}} = 1,$$

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* The problem of expressing the n^2 direction-cosines in terms of $\frac{n \cdot n-1}{2}$ essential parameters has been solved by Cayley, Crelle, 32 (1846). See also Klein, "Nicht Euclidische Geometrie," II, p. 109 (Vorlesungshefte).

and all the other coefficients equal to zero. We then obtain the following simple equations

$$\left.\begin{array}{llll} \dfrac{\delta x_1}{\delta t} = x_2 \quad , & \dfrac{\delta x_2}{\delta t} = -x_1 \quad , & \dfrac{\delta x_3}{\delta t} = x_4 , & \dfrac{\delta x_4}{\delta t} = -x_3 , \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{\delta x_{n-2}}{\delta t} = x_{n-1} , & \dfrac{\delta x_{n-1}}{\delta t} = -x_{n-2} , & \dfrac{\delta x_n}{\delta t} = C_n , & \end{array}\right\} \qquad (12'')$$

and we have the

THEOREM.—*The most general infinitesimal motion in space of n dimensions (n odd) consists of a translation along an axis and* $\dfrac{n-1}{2}$ *rotations.* We shall call such a motion *an n-dimensional screw motion.**

Following S. Lie's method† we shall define a line-element in the space M_n as a given point and a direction through it, so that a line-element is completely determined when the coordinates of the point and the direction-cosines, or, what is the same thing, quantities proportional to these are given, that is to say, when the quantities $x_1, x_2 \ldots . x_n, dx_1 : dx_2 : \ldots . : dx_n$ are given.

An infinitesimal screw-motion will of course also transform the line-elements of which through each point pass ∞^{n-1}. We shall now consider all those line-elements that are perpendicular to the direction through the point when it is subjected to a screw-motion. For all such elements the following relation must hold

$$dx_1\delta x_1 + dx_2\delta x_2 + \ldots . + dx_n\delta x_n = 0 ,$$

which becomes after substituting the values of $\delta x_1, \delta x_2, \ldots . , \delta x_n$ obtained from (12'')

$$x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 + \ldots . + x_{n-1}dx_{n-2} - x_{n-2}dx_{n-1} + C_ndx_n = 0 . \qquad (15)$$

Since there are ∞^{2n-1} line-elements there will be ∞^{2n-2} of these satisfying the above differential equation.

Let us inquire what lines satisfy the equation. We write any given line in the form

$$x_i = \rho_i x_n + \sigma_i , \quad (i = 1, 2, \ldots . , n-1) \qquad (16)$$

* For a discussion of infinitesimal motion in the space (x, y, z) see S. Lie, Geom. der Berührungstr. Vol. I, pp. 206–212.

† S. Lie, Ibid. p. 11.

Substituting in the equation (15) we obtain the following relation between the parameters

$$\sum_{i=1}^{i=\frac{n-1}{2}} (\sigma_{2i}\rho_{2i-1} - \sigma_{2i-1}\rho_{2i}) + C_n = 0. \tag{17}$$

Now, since there are ∞^{2n-2} lines, ∞^{2n-3} of these will satisfy the differential equation (15), namely, all those lines whose parameters satisfy the relation (17). We have then the following extension of a well-known theorem in kinematics of three dimensions:

By an infinitesimal n-dimensional screw-motion, there exist ∞^{2n-3} lines whose points move in a direction perpendicular to the direction of the motion.

Through each point in M_n pass ∞^{n-3} such lines which aggregate we may call a *pencil;* of such pencils, there are in M_n ∞^n and the aggregate of all ∞^{2n-3} lines we shall call a *Nullsystem.*

II.

1. The case $n = 5$ is of special interest, as the following development will show. Our differential equation reduces to

$$x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 + C_5dx_5 = 0.$$

Putting $C_5dx_5 = dx_5'$, $x_1 = x_1'$, $x_2 = x_2'$, $x_3 + x_3'$, $x_4 = x_4'$, this equation becomes

$$x_2'dx_1' - x_3'dx_2' + x_4'dx_3' - x_3'dx_4' + dx_5 = 0, \tag{1}$$

and the relation (14) reduces to

$$\rho_2\sigma_1 - \rho_1\sigma_2 + \rho_4\sigma_3 - \rho_3\sigma_4 - 1 = 0.$$

Before we proceed any further, we shall introduce a few definitions due to Lie.*

A surface-element in ordinary space consists of a point x_1, x_2, x_3 and a plane passing through it. Since the direction-cosines of the plane are determined by the quantities $p_1 = \dfrac{\partial x_3}{\partial x_1}$, $p_2 = \dfrac{\partial x_3}{\partial x_2}$ and -1, we may consider x_1, x_2, x_3, p_1, p_2 as the coordinates of an element. There exist in space ∞^5 such elements, so that ordinary space may be considered as a five-dimensional manifoldness, if we choose for space-elements all the ∞^5 surface-elements. A family of surface-

* Lie-Scheffer, Berührungstr., p. 523.

elements may be expressed analytically by an equation

$$F(x_1, x_2, x_3, p_1, p_2) = 0 .$$

Two surface-elements are infinitely near each other when the coordinates of one element differ by an infinitesimal amount from those of the other; that is to say, they are determined by the coordinates $(x_1, x_2, x_3, p_1, p_2)$ and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, p_1 + dp_1, p_2 + dp_2, p_3 + dp_3)$. Whenever the point of the latter element lies in the plane of the former, the two elements are said to have *united position* (Vereinigte Lage). The analytical condition for this may easily be proved to be

$$dx_3 - p_1 dx_1 - p_2 dx_2 = 0 .$$

An aggregate of surface-elements, in which each element has united position with all the elements infinitely near it, is called an element-manifoldness, or, shortly, an element-M. (Element-Verein).

If, now, we employ the transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_5 + x_1 x_2 + x_3 x_4 = X_3,$$

where X_1, X_2, X_3, P_1, P_2 represent the coordinates of a surface-element in M_3 and x_1, x_2, x_3, x_4, x_5 those of a point in the space M_5, a one-to-one correspondence is established between all the ∞^5 surface-elements in M_3 and the ∞^5 points of M_5 This transformation is due to Lie.*

Equation (1) may now be written

$$d(x_5 + x_1 x_2 + x_3 x_4) - 2x_1 dx_2 - 2x_3 dx_2 = 0 , \tag{2}$$

or, in terms of the coordinates of M_3,

$$dX_3 - P_1 dX_1 - P_2 dX_2 = 0 . \tag{2'}$$

Hence we conclude

To a point of M_5 satisfying the Pfaffian equation (2) there corresponds in M_3 a surface-element; and to a point-manifoldness in M_5 there will correspond in M_3 an element-M.

The simplest kind of element-M in M_3 consists of a point $X_1 = a$, $X_2 = b$, $X_3 = c$ and all the ∞^2 planes passing through it. To this there corresponds in

* Lie, Theorie der Transformationsgruppen, 2te Absch., p. 521.

M_5 the two-dimensional plane

$$x_2 = a, \quad x_4 = b, \quad x_5 + ax_1 + bx_3 = c.$$

If the element-M be defined by the four equations

$$X_1 = a, \quad X_2 = b, \quad X_3 = c, \quad \phi(P_1, P_2) = 0,$$

we have what is called an element-cone, consisting of a point and ∞^1 planes passing through it which are tangent to the cone having the point as vertex. In the space M_5 we obtain a curve given by the equations

$$x_2 = a, \quad x_4 = b, \quad x_5 + ax_1 + bx_3 = c, \quad \phi(2x_1, 2x_3) = 0.$$

Again, suppose the element-M be defined by the equations

$$\omega_1(X_1, X_2, X_3) = 0, \quad \omega_2(X_1, X_2, X_3) = 0;$$

solving for X_1 and X_2 we obtain

$$X_1 = \xi_1(X_3) \qquad X_2 = \xi_2(X_3) \tag{3}$$

and since the differential equation (2′) must be satisfied by all the surface-elements, we must also have

$$1 - \frac{d\xi_1}{dX_3} P_1 - \frac{d\xi_2}{dX_3} P_2 = 0. \tag{4}$$

Equations (3) and (4) define ∞^2 surface-elements of a curve to which in M_5 there corresponds the two-dimensional surface

$$x_2 = \xi_1(x_5 + x_1x_2 + x_3x_4), \quad x_4 = \xi_2(x_5 + x_1x_2 + x_3x_4),$$
$$1 - 2x_1\left[\frac{d\xi_1}{dX_3}\right] - 2x_3\left[\frac{d\xi_2}{dX_3}\right] = 0,$$

where the bracketed derivatives stand for what is obtained after substituting for X_3 its value in terms of the coordinates of M_5. If to the equations (3) and (4) we add a fourth containing besides the X's also P_1 and P_2, say

$$\omega_3(P_1, P_2, X_1, X_2, X_3) = 0,$$

we obtain an element-M containing only ∞^1 surface-elements. Let this equation be put in the form

$$\rho(X_3, P_1, P_2) = 0. \tag{5}$$

Solving (4) and (5) for P_1 and P_2 we put

$$P_1 = \xi_3(X_3), \quad P_2 = \xi_4(X_3); \tag{6}$$

these values of P_1 and P_2 must satisfy (4), that is, we must have

$$1-\frac{\partial \xi_1}{\partial X_3}\xi_3-\frac{\partial \xi_2}{\partial X_3}\xi_4=0,$$

which may always be satisfied for arbitrary values of ξ_1 and ξ_2 by choosing suitable values for ξ_3 and ξ_4. The ∞^1 surface-elements defined by these equations have their planes tangents at each point along the curve. Such an element-M we shall call an *element-band* (Element-Streife). In the space M_5 there corresponds to this element-M a curve.

Suppose, finally, the element-M be defined by a single equation

$$\lambda(X_1, X_2, X_3)=0; \tag{7}$$

solving for X_3, we put

$$X_3=\eta(X_1, X_2),$$

and, since the equation (4) must also be satisfied, we must have

$$P_1=\frac{\partial \eta}{\partial X_1}, \quad P_2=\frac{\partial \eta}{\partial X_2}. \tag{8}$$

The equations (7) and (8) define ∞^2 surface-elements of a surface whose transform in M_5 is a two-dimensional surface. We shall now summarize these results in the following table:

Space M_3.	Space M_5.
(1). $dX_3-P_1dX_1-P_2dX_2=0$.	(1). $dx_5+x_2dx_1-x_1dx_2 + x_4dx_3-x_3dx_4=0$.
(2). ∞^2 surface-elements of a point.	(2). A two-dimensional plane.
(3). ∞^1 surface-elements of a point or element-cone.	(3). A plane curve.
(4). ∞^2 surface-elements of a curve.	(4). A two-dimensional surface.
(5). ∞^1 surface-elements of a curve or element-band.	(5). A curve.
(6). ∞^2 surface-elements of a surface.	(6). A two-dimensional surface.

2. We shall now resume the study of the nullsystem in M_5, and we propose to find the element-M in M_3 corresponding to an ensemble of points represented

by all the lines of the nullsystem. For this purpose it will be convenient to put the nullsystem in the form

$$x_i = \rho_i t + \sigma_i, \qquad (i = 1, 2, 3, 4, 5) \tag{1}$$

in which $\rho_5 = \rho_2\sigma_1 - \sigma_2\rho_1 + \sigma_3\rho_4 - \sigma_4\rho_3$. Introducing the new coordinates from the transformation on page 114, we have

$$\left.\begin{aligned} &\frac{P_1}{2} = \rho_1 t + \sigma_1, \quad \frac{P_2}{2} = \rho_3 t + \sigma_3, \\ &X_1 = \rho_2 t + \sigma_2, \quad X_2 = \rho_4 t + \sigma_4, \\ &X_3 = (\rho_1\rho_2 + \rho_3\rho_4)\, t^2 + 2(\rho_2\sigma_1 + \rho_4\sigma_3)\, t + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5. \end{aligned}\right\} \tag{2}$$

Does this system define an element-M in M_3? To answer this question we proceed as follows: eliminating t from the above equations, we obtain

$$\left.\begin{aligned} &\text{(a)} \quad \rho_2 \frac{P_1}{2} = \rho_1 X_1 + \sigma_1\rho_2 - \sigma_2\rho_1, \\ &\text{(b)} \quad \rho_2 \frac{P_2}{2} = \rho_3 X_1 + \sigma_3\rho_2 - \sigma_2\rho_3, \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} &\text{(a)} \quad \rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2, \\ &\text{(b)} \quad X_3 = \frac{(\rho_1\rho_2 + \rho_3\rho_4)}{\rho_2^2}(X_1 - \sigma_2)^2 + 2\frac{(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5. \end{aligned}\right\} \tag{4}$$

Multiplying the first equation by ρ_3 and the second by ρ_1 and subtracting we obtain

$$\text{(c)} \quad \rho_3 \frac{P_1}{2} - \rho_1 \frac{P_2}{2} - \sigma_1\rho_3 + \sigma_3\rho_1 = 0. \tag{4}$$

If now equations [4] (a) and (b) define an element-M we must have

$$dX_3 - P_1 dX_1 - P_2 dX_2 \equiv 0$$

by virtue of the above system, that is to say, the equation

$$\rho_2 P_1 + \rho_4 P_2 - \frac{2(\rho_1\rho_2 + \rho_3\rho_4)}{\rho_2}(X_1 - \sigma_2) - 2(\rho_2\sigma_1 + \rho_4\sigma_2) = 0$$

must be satisfied for values of P_1 and P_2 derived from equations [3] (a) and (b) which is easily seen to be the case. Q. E. D. We have then the

THEOREM.—*To the* ∞^1 *points of a line belonging to a nullsystem in* M_5 *there*

corresponds in M_3 by virtue of the transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_1x_2 + x_3x_4 + x_5 = X_3$$

∞^1 surface-elements forming an element-M. This element-M is formed by the curve defined by the equations

$$\rho_4 X_1 - \rho_2 X_2 - \sigma_2\rho_4 + \sigma_4\rho_2 = 0,$$

$$\frac{(\rho_1\rho_2 + \rho_3\rho_4)}{\rho_2^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5 - X_3 = 0,$$

and all the ∞^1 tangent planes along it. The element-M is therefore an *element-band.* The curve may be considered as the intersection of a plane parallel to the X_3-axis and a parabolic cylinder parallel to the X_2-axis. Hence we may say

To the ∞^1 points of a line belonging to a nullsystem in M_5 there corresponds in M_3 ∞^1 surface-elements forming an element-band whose point-locus is a parabola having its plane parallel to the X_3-axis.

To a fixed line of the nullsystem there corresponds one and only one element-band, but it does not follow that to an element-band corresponds only one line of the nullsystem. In fact we shall prove that the correspondence between these space-elements is not a one-to-one. Let us first study the element-band considered as a point-locus merely. We write the nullsystem in the modified form

$$\left.\begin{array}{ll} x_1 = \rho_1' x_5 + \sigma_1, & x_2 = \rho_2' x_5 + \sigma_2, \\ x_3 = \rho_3' x_5 + \sigma_3, & x_4 = \rho_4' x_5 + \sigma_4, \end{array}\right\} \tag{5}$$

which is obtained from (1) by putting $\sigma_5 = 0$ and $\rho_i' = \frac{\rho_i}{\rho_5}$, $(i = 1, 2, 3, 4)$ and then eliminating the parameter t. The condition that (5) shall be a nullsystem must now be written

$$1 = \rho_2'\sigma_1 - \rho_1'\sigma_2 + \rho_4'\sigma_3 - \rho_3'\sigma_4, \tag{6}$$

while the equations (4), (a) and (b), take the form

$$\rho_4' X_1 - \rho_2' X_2 - \sigma_2\rho_4' + \sigma_4\rho_2' = 0,$$

$$\frac{(\rho_1'\rho_2' + \rho_3'\rho_4')}{\rho_2'^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2'\sigma_1 + \rho_4'\sigma_3)}{\rho_2'}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 - X_3 = 0.$$

There exist in M_3, ∞^5 such parabolae whose parameters may evidently be written

$$K = \frac{\rho_2'}{\rho_4'}, \qquad L = \sigma_2 - \sigma_4 K,$$

$$A = \frac{\rho_1'\rho_2' + \rho_3'\rho_4'}{\rho_2'^2}, \quad B = \frac{2(\rho_2'\sigma_1 + \rho_4'\sigma_3)}{\rho_2'} - 2\sigma_2 A,$$

$$C = \sigma_1\sigma_2 + \sigma_3\sigma_4 - 2\sigma_2\frac{(\rho_2'\sigma_1 + \rho_4'\sigma_3)}{\rho_2'} + \sigma_2^2 A.$$

Consider now any fixed curve, that is, let A, B, C, K and L be fixed quantities and solve the above equations for ρ_1', ρ_2', ρ_3', σ_1, σ_2 and σ_3, taking also account of the relation (6). We obtain the following set of equations:

$$\left.\begin{aligned}
&\rho_2' = K\rho_4', && \sigma_1 = B + AL + \sigma_4\left[AK + \frac{BK}{2L}\right] + \frac{C}{L},\\
&\rho_1' = \frac{K}{L}\left[AL + \frac{B}{2}\right]\rho_4' - \frac{1}{L}, && \sigma_2 = L + \sigma_4 K,\\
&\rho_3' = \frac{K}{L}\left[1 - \rho_4'\frac{KB}{2}\right], && \sigma_3 = -\frac{BK}{2}\left[1 + \frac{\sigma_4 K}{L}\right] - \frac{CK}{L}.
\end{aligned}\right\} \quad (7)$$

It follows from these equations that to a fixed parabola in M_3 there corresponds not only one, but ∞^2 lines of the nullsystem, a set of ∞^1 lines being obtained by letting ρ_4' vary and another set of ∞^1 lines by letting σ_4 vary. *Each parabola must therefore be considered as a bundle of ∞^2 coincident curves.*

Remark. A system of lines may be chosen from among the lines of the nullsystem in such a way as to make the correspondence between the configurations a one-to-one. We may call such a system a *one-to-one* system; it may be determined by two relations, one between the ρ's and one between the σ's, say

$$\omega_1(\rho_1', \rho_2', \rho_3', \rho_4') = 0, \qquad \omega_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = 0,$$

subject to the restriction that none of the parameters A, B, C, K, L shall vanish by virtue of the given relations, or become expressible in terms of one or more of the others.

It now remains to investigate the curve itself considered as an element-band. The tangent plane at a fixed point (X_1, X_2, X_3) has for direction-cosines

quantities proportional to P_1, P_2, -1, P_1 and P_2 being determined by the relations

$$\left.\begin{aligned} \rho_2' \frac{P_1}{2} &= \rho_1' X_1 + \sigma_1\rho_2' - \sigma_2\rho_1', \\ \rho_4' \frac{P_2}{2} &= \rho_3' X_2 + \sigma_3\rho_4' - \sigma_4\rho_3'. \end{aligned}\right\} \qquad (4)\,(c)$$

Now these equations show that there can only be ∞^1 surface-elements associated with each point on the curve. In fact, expressing ρ_1', ρ_2', ρ_3' in terms of ρ_4' by means of equations (7) and then eliminating this least parameter from the two equations(4)(c), we obtain a relation between P_1 and P_2 depending on the parameters σ_1, σ_2, σ_3 and σ_4, of which the three first are functions of the last by virtue of the equations (7). We may, therefore, conceive of the configuration as consisting of ∞^1 element-bands having a single parabola as point-locus, or, in other words, it will consist of all the ∞^2 surface-elements of the parabola. We may state the results obtained thus:

To all the ∞^7 *lines of the nullsystem* (1) *in the space* M_5 *there correspond* ∞^5 *parabolae formed by the intersection of the planes*

$$\rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2 \qquad (8)$$

and the parabolic cylinders

$$X_3 = \frac{\rho_1\rho_2 + \rho_3\rho}{\rho_2^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5. \qquad (9)$$

These parabolae must be considered each as a bundle of ∞^2 *coincident ones having their surface-elements united into a set of* ∞^1 *element-bands, so that the configuration corresponding to the nullsystem consists of* ∞^5 *parabolae* (8) *and* (9) *and the* ∞^5 *surface-elements of each parabola.*

The question now suggests itself: How are the ∞^2 lines of the nullsystem corresponding to a single parabola distributed in the space M_5? Introducing the new parameters A, B, C, K, L, ρ_4', σ into our nullsystem it takes the form

$$\left.\begin{aligned} x_1 &= \left[\left(KA + \frac{BK}{2L}\right)\rho_4' - \frac{1}{L}\right]x_5 + B + AL + \frac{C}{L} + \sigma_4\left(AK + \frac{BK}{2L}\right), \\ x_2 &= K\rho_4' x_5 + L + \sigma_4 K, \\ x_3 &= \frac{K}{L}\left(1 - \rho_4'\frac{KB}{2}\right)x_5 - \frac{B}{2}\left(K + \sigma_4\frac{K^2}{L}\right) - \frac{CK}{L}. \\ x_4 &= \rho_4' x_5 + \sigma_4. \end{aligned}\right\} \qquad (10)$$

If now we consider ρ_4', σ_4 as variable parameters while the others are fixed, we obtain all the ∞^2 lines of the nullsystem corresponding to a given parabola in M_3. These lines lie in the two-dimensional plane defined by the three flat spaces or lineoids

$$\left.\begin{aligned} & x_2 - Kx_4 - L = 0\,, \\ & x_3 + \frac{BK^2}{2L} x_4 - \frac{K}{L} x_5 + \frac{BK}{2} + \frac{CK}{L} = 0\,, \\ & x_1 - \left(AK + \frac{BK}{2L}\right) x_4 + \frac{x_5}{L} - B - AL - \frac{C}{L} = 0\,, \end{aligned}\right\} \tag{11}$$

obtained by eliminating ρ_4' and σ_4 from the system (10). There are in M_5 ∞^5 such planes, corresponding to the ∞^5 parabolae in M_3. Through each point in M_5 pass ∞^2 such planes and hence through each point of M_3 considered as a surface-element there must pass ∞^2 parabolae; (this is also evident from the consideration that there are ∞^2 parabolae in space having their planes parallel to the X_3-axis and being also tangent to a given plane at a given point in the plane.) But we know that through a given point in M_5 pass ∞^3 lines of the nullsystem; hence these lines must be distributed in the ∞^2 planes passing through the point, ∞^1 of them being situated in each plane and passing through the point. We have thus arrived at a definite idea of the grouping of the lines of the nullsystem in M_5, the transformation into ordinary space having furnished us a means of geometrical "Anschauung". The two-dimensional manifoldness (11) in which are situated all the ∞^2 lines of the nullsystem corresponding to a given parabola in M_3 we shall call *a point-M_2*, and we may state the result obtained this:

To all the lines of a point-M_2 in M_5 there corresponds in M_3 a parabola, and conversely, to a parabola in M_3 there corresponds all the lines of a point-M_2.

We may now collect these results in the following table:

Space M_5.	Space M_3.
(1). $dx_5 + x_2 dx_1 - \dots = 0$.	(1). $dX_3 - P_1 dX_3 - P_2 dX_2 = 0$.
(2). Point.	(2). Surface-element.
(3). A straight line of the nullsystem.	(3). A parabola considered as an element-band.
(4). All the ∞^2 lines of an element-M_2.	(4). A single parabola considered as ∞^1 united element-bands.
(5). All the ∞^3 lines of the nullsystem passing through a fixed point.	(5). ∞^2 parabolae passing through a fixed point.
(6). All the ∞^2 point-M_2's passing through a fixed point.	(6). ∞^2 parabolae passing through a fixed point.
(7). All the ∞^5 point-M_2's.	(7). All the ∞^5 parabolae.

3. If we choose all the lines of the nullsystem for which $\rho_1\rho_2 + \rho_3\rho_4 = 0$ we obtain the ∞^4 straight lines

$$\left.\begin{aligned} X_1 &= \rho_2 t + \sigma_2 , \\ X_2 &= \rho_4 t + \sigma_4 , \\ X_3 &= 2(\rho_2\sigma_1 + \rho_4\sigma_3)\, t + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5 , \end{aligned}\right\} \tag{1}$$

the planes of whose surface-elements are determined by the equations

$$\frac{P_1}{2} = \rho_1 t + \sigma_1 ,$$

$$\frac{P_2}{2} = \rho_3 t + \sigma_3 ,$$

from which we obtain by virtue of the above relations between the ρ's

$$\rho_2 \frac{P_1}{2} + \rho_4 \frac{P_2}{2} = \sigma_1\rho_2 + \sigma_3\rho_4 . \tag{2}$$

Now this relation is identical with the relation (3) of page 117; in fact, multiplying it by ρ_3 and (3) by ρ_2, we get by adding

$$(\rho_1\rho_2 + \rho_3\rho_4)\frac{P_2}{2} \equiv \sigma_1(\rho_2\rho_3 - \rho_2\rho_3) + \sigma_3(\rho_1\rho_2 + \rho_3\rho_4) \equiv 0 .$$

Hence, only one relation exists between P_1 and P_2. We may therefore say:

To all the ∞^6 lines of M_5 satisfying the differential equations

$$\left.\begin{aligned} dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 &= 0\,, \\ dx_1dx_2 + dx_3dx_4 &= 0 \end{aligned}\right\} \tag{3}$$

there correspond in M_3 all the ∞^4 lines of that space, together with all the surface-elements of these lines.

Eliminating the parameter t from the equation (1), we obtain

$$X_1 = r_1X_3 + s_1, \quad X_2 = r_2X_3 + s_2\,,$$

where

$$r_1 = \frac{\rho_2}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,, \quad r_2 = \frac{\rho_4}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,,$$

$$s_1 = \frac{2\sigma_2(\rho_2\sigma_1 + \rho_4\sigma_3) - \rho_2(\sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5)}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,,$$

$$s_2 = \frac{2\sigma_4(\rho_2\sigma_1 + \rho_4\sigma_3) - \rho_4(\sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5)}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,.$$

The complex of lines satisfying the differential equations (3) is thus seen to be a remarkable one, inasmuch as a correspondence exists between all the lines belonging to it and all the lines of ordinary space, these lines being considered as an aggregate of surface-elements. The ∞^6 lines of the complex satisfy a certain geometrical condition; in fact, they are lines belonging to the system of ∞^4 4-dimensional cylinders of the second degree

$$(x - \sigma_1)(x_2 - \sigma_2) + (x_3 - \sigma_3)(x_4 - \sigma_4) = 0$$

parallel to the x_5-axis. A complex of this kind we shall call *asymptotic.* The reason for this name will appear later. Its equations are

$$\left.\begin{aligned} x_1 &= \frac{1}{L}\left(\frac{BK}{2}\rho_4' - 1\right)x_5 + B + \frac{C}{L} + \sigma_4\frac{BK}{2L}\,, \\ x_2 &= K\rho_4'x_5 + L + \sigma_4K\,, \\ x_3 &= \frac{K}{L}\left(1 - \rho_4'\frac{KB}{2}\right)x_5 - \frac{B}{2}\left(K + \sigma_4\frac{K^2}{L}\right) - \frac{CK}{L}\,, \\ x_4 &= \rho_4'x_5 + \sigma_4. \end{aligned}\right\} \tag{4}$$

To a line in M_5 corresponds a certain line in M_3 considered as an element-band, which may be looked upon as a line with a plane passing through it; but to a line in M_3 there will correspond ∞^2 lines of the nullsystem which lie in the two-dimensional manifoldness defined by the equations

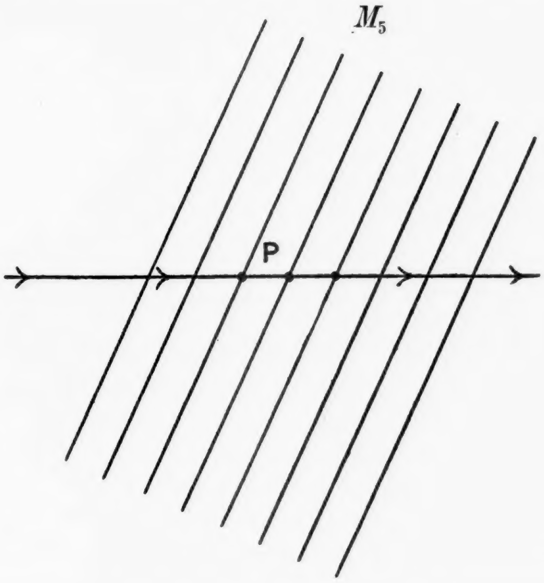
$$\begin{aligned}
&x_2 - \quad Kx_4 - L = 0\,,\\
&x_1 - \frac{BK}{2L}x_4 + \frac{x_5}{L} - B - \frac{C}{L} = 0\,,\\
&\frac{x_3}{K} + \frac{BK}{2L}x_4 - \frac{x_5}{L} + \frac{B}{2}\left(1 + \frac{2C}{BL}\right).
\end{aligned}$$

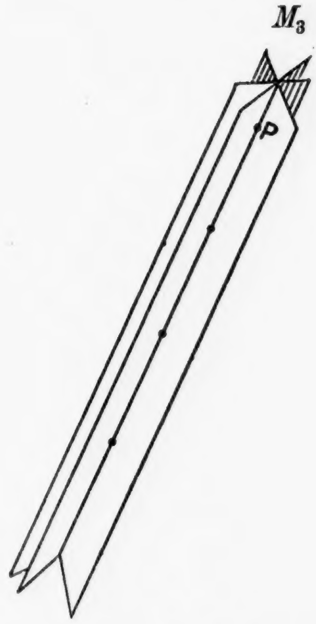
Through a given point in M_5 there pass ∞^1 such point-M_2's, hence, through a fixed point in M_3 pass ∞^1 lines all lying in the plane of the surface-element determined by the coordinates of the fixed point in M_5. To all the ∞^1 lines of the nullsystem passing through a given point and lying in a given point-M_2 corresponds a single line in M_3 passing through a point and lying in a plane whose coordinates P_1 and P_2 are determined by the corresponding values of x_1 and x_3, (the plane of the surface-element). Suppose now that the point moves in any given point-M_2; the corresponding point in M_3 will move along the line, while the plane of the surface-element of the line will turn around the line as an axis. The reason why we only get ∞^1 planes, while there are ∞^2 points in the point-M_2 is explained by the fact that there are ∞^1 points in it that determine the same plane, namely all the points of the point-M_2 for which $x_1 = \frac{P_1}{2} = \text{const.}$ $x_3 = \frac{P_2}{2} = \text{const.}$ These points lie on a certain line whose equations are easily obtained, viz.,

$$\left.\begin{aligned}
x_1 &= C_1,\\
x_2 &= Kx_4 + L,\\
x_3 &= -K\left[C_1 - \frac{B}{2}\right],\\
x_4 &= \frac{2}{BK}x_5 + \left(C_1 - B - \frac{C}{L}\right)\frac{2L}{BK}.
\end{aligned}\right\} \tag{5}$$

Now, suppose C_1 a variable parameter; we obtain ∞^1 parallel lines in the point-M_2, to each of which there corresponds in M_3 a different plane. If, then, the point moves along some line cutting this set of parallel lines at some angle, the plane of the surface-element will rotate around the line as an axis and com-

plete one revolution when the point P has returned to its original position. In the same way, if one of the lines (5) move parallel to itself, the plane of the surface-element in M_3 will rotate and resume its original position when the line has returned to its original position. Hence, it follows that there is a one-to-one correspondence between the ∞^1 lines (5) and the ∞^1 planes of the surface-elements of a line in M_3.





The following table will now be convenient as a résumé of the results obtained:

Space M_5.	Space M_3.
(1). $dx_5 + x_2dx_1 - x_1dx_2 + x_4dx^3 - x_3dx_4 = 0$.	(1). $dX_3 - P_1dX_1 - P_2dX_2 = 0$.
(2). $dx_1dx_2 + dx_3dx_4 = 0$.	(2). $dP_1dX_1 + dP_2dX_2 = 0$.
(3). Point.	(3). Surface-element.
(4). All the ∞^6 lines of the complex.	(4). All the ∞^4 lines of M_3 and the ∞^1 surface-elements of each line.
(5). The ∞^2 lines of a point-M_2.	(5). A single line and its ∞^1 surface-elements.
(6). All the ∞^1 point-M_2's passing through a fixed point.	(6). All the ∞^1 lines passing through a fixed point and lying in the plane of the surface-element determined by the point.
(7). All the ∞^1 parallel lines (5) lying in a point-M_2.	(7). A line and its ∞^1 surface-elements.

4. The study of geometrical relations in hyper-space is often very useful for the investigation of geometrical relations in ordinary space. If a correspondence of a higher space with ordinary space can be effected by means of a transformation, we are very often led to interesting properties of space that would otherwise not have been so evident.

In the following we shall show how the study of an asymptotic complex will lead to the study of all surfaces whose asymptotic lines are known.

A curve c_5 in the space M_5 is said to be a curve of the nullsystem whenever the linear tangent at each point of the curve belongs to the nullsystem. The most general curve of the system has the form

$$x_1 = \phi_1(u), \quad x_2 = \phi_2(u), \quad x_3 = \phi_3(u), \quad x_4 = \phi_4(u),$$
$$x_5 = \int (\phi_1\phi_2' - \phi_2\phi_1' + \phi_3\phi_4' - \phi_4\phi_3')\, du.$$

In M_3 we get a curve c_3,

$$X_1 = \phi_2, \qquad X_2 = \phi_4,$$
$$X_3 = \phi_1\phi_2 + \phi_3\phi_4 + \int (\phi_1\phi_2' - \phi_2\phi_1' + \phi_3\phi_4' - \phi_4\phi_3')\, du.$$

At each point of this curve, the plane of the surface-elements is tangent to the curve. To the curve c_5, considered as the envelope of all its tangents, corresponds the curve c_3 considered as the envelope of ∞^1 parabolae. Now, suppose that the nullsystem is asymptotic; we have then in addition the relation

$$\phi_1'\phi_2' + \phi_3'\phi_4' = 0,$$

and our curve must have the form

$$x_1 = \phi_1, \quad x_2 = \phi_2, \quad x_3 = \phi_3, \quad x_4 = -\int \frac{\phi_1'\phi_2'}{\phi_3'}\, du,$$
$$x_5 = \int \left[\phi_1\phi_2' - \phi_2\phi_1' - \phi_3 \cdot \frac{\phi_1'\phi_2'}{\phi_3'} + \phi_3' \cdot \int \frac{\phi_1'\phi_2'}{\phi_3'}\, du \right] du,$$

to which there corresponds the following curve in M_3:

$$X_1 = \phi_2, \quad X_2 = \phi_4 - \int \frac{\phi_1'\phi_2'}{\phi_3'} du, \quad X_3 = \phi_1\phi_2 - \phi_3 \int \frac{\phi_1'\phi_2'}{\phi_3'} du + \phi_5,$$

where $\phi_5 = x_5$. *The plane of the surface-element at each point of this curve osculates the curve.* In fact, we have

$$\frac{dx_5}{du} - x_1\frac{dx_2}{du} + x_2\frac{dx_1}{du} - x_3\frac{dx_4}{du} - x_4\frac{dx_3}{du} = \frac{dX_3}{du} - P_1\frac{dX_1}{du} - P_2\frac{dX_2}{du} = 0,$$

$$\frac{dx_1}{du}\frac{dx_2}{du} + \frac{dx_3}{du}\frac{dx_4}{du} = \frac{dX_1}{du}\frac{dP_1}{du} + \frac{dX_2}{du}\frac{dP_2}{du} = 0;$$

of these equations, the first expresses the fact that the plane of the surface-element is tangent to the curve, while the second is nothing but the condition that the tangent plane shall osculate the curve. Q. E. D.

A two-dimensional surface S_2 in M_5 is said to belong to the nullsystem whenever the curvilinear coordinate system $u = \text{const.}$, $v = \text{const.}$, belongs to the system. Let the system be determined by the equations

$$X_i = \phi_i(u, v). \qquad (i = 1, 2, 3, 4, 5) \qquad (1)$$

In order that this system shall belong to the nullsystem, the following conditions must be satisfied:

$$\left.\begin{aligned} \frac{\partial\phi_5}{\partial u} &= \phi_1\frac{\partial\phi_2}{\partial u} - \phi_2\frac{\partial\phi_1}{\partial u} + \phi_3\frac{\partial\phi_4}{\partial u} - \phi_4\frac{\partial\phi_3}{\partial u}, \\ \frac{\partial\phi_5}{\partial v} &= \phi_1\frac{\partial\phi_2}{\partial v} - \phi_2\frac{\partial\phi_1}{\partial v} + \phi_3\frac{\partial\phi_4}{\partial v} - \phi_4\frac{\partial\phi_3}{\partial v}. \end{aligned}\right\} \qquad (2)$$

Now, since $d\phi_5$ must be a perfect differential, we have

$$\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial u} - \frac{\partial\phi_2}{\partial v}\frac{\partial\phi_1}{\partial u} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial u} - \frac{\partial\phi_4}{\partial v}\frac{\partial\phi_3}{\partial u} = 0. \qquad (3)$$

Treating this equation as a linear differential equation in $\frac{\partial\phi_2}{\partial u}$ and $\frac{\partial\phi_2}{\partial v}$, it may be written

$$A\frac{\partial\phi_2}{\partial u} - B\frac{\partial\phi_2}{\partial v} = C,$$

where $$A = \frac{\partial\phi_1}{\partial v}, \quad B = \frac{\partial\phi_1}{\partial u}, \quad C = \frac{\partial\phi_4}{\partial v}\frac{\partial\phi_3}{\partial u} - \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial u}.$$

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We find, by integrating,

$$\phi_1(u, v) = c_1, \quad \phi_2 = -\int \left[\frac{C}{B}\right]_{u=\rho(v, c)} dv = \xi(v, c_1) + c_2,$$

where the expression $\left[\frac{C}{B}\right]$ stands for the value of the function $\frac{C}{B}$ after substituting in it for u the value of $\rho(v)$ obtained by solving $\phi_1(u, v) = c_1$ for u. The general integral of (3) is, therefore,

$$\phi_2 = \Phi(\phi_1) + \xi(v, \phi(u, v)),$$

where the function Φ is arbitrary. We have then the following

THEOREM.—*The most general two-dimensional surface in M_5 belonging to a nullsystem is given by the equations*

$$\left.\begin{aligned} x_1 &= \phi_1(u, v), \\ x_2 &= \phi_2(u, v) = \Phi(\phi_1) - \int \left[\frac{C}{B}\right]_{u=\rho(v, c_1)} dv = \Phi(\phi_1) + \xi(v, \phi_1(u, v)), \\ x_3 &= \phi_3(u, v), \\ x_4 &= \phi_4(u, v), \\ x_5 &= \phi_5(u, v) = \int \frac{\partial \phi_5}{\partial u} du + \frac{\partial \phi_5}{\partial v} dv, \end{aligned}\right\} \tag{4}$$

where

$$\frac{C}{B} = \frac{\dfrac{\partial \phi_4}{\partial v}\dfrac{\partial \phi_3}{\partial u} - \dfrac{\partial \phi_3}{\partial v}\dfrac{\partial \phi_4}{\partial u}}{\dfrac{\partial \phi_1}{\partial u}}$$

and ϕ_5 satisfies the conditions

$$\frac{\partial \phi_5}{\partial u} = \phi_1 \frac{\partial \phi_2}{\partial u} - \phi_2 \frac{\partial \phi_1}{\partial u} + \phi_3 \frac{\partial \phi_4}{\partial u} - \phi_4 \frac{\partial \phi_3}{\partial u},$$

$$\frac{\partial \phi_5}{\partial v} = \phi_1 \frac{\partial \phi_2}{\partial v} - \phi_2 \frac{\partial \phi_1}{\partial v} + \phi_3 \frac{\partial \phi_4}{\partial v} - \phi_4 \frac{\partial \phi_3}{\partial v}.$$

The corresponding surface in M_3 may now be written

$$\left.\begin{aligned} X_1 &= \phi_2(u, v) = \Phi(\phi_1) + \xi(v, \phi_1(u, v)), \\ X_2 &= \phi_4(u, v), \\ X_3 &= \phi_1(u, v)\phi_2(u, v) + \phi_3(u, v)\phi_4(u, v) + \phi_5(u, v). \end{aligned}\right\} \tag{5}$$

To a point $u = \text{const.}$, $v = \text{const.}$ on the surface (4) there corresponds on (5) a surface-element consisting of a point $u = \text{const.}$, $v = \text{const.}$ and a plane passing through it. *This plane is tangent to the surface at that point.* In fact, the condi-

tions that the plane shall be tangent to the surface are

$$\left.\begin{aligned} \frac{\partial X_3}{\partial u} - P_1\frac{\partial X_1}{\partial u} - P_2\frac{\partial X_2}{\partial u} = 0,\\ \frac{\partial X_3}{\partial v} - P_1\frac{\partial X_1}{\partial v} - P_2\frac{\partial X_2}{\partial v} = 0,\end{aligned}\right\} \tag{6}$$

and if we calculate the partial derivatives of X_1, X_2 and X_3, as well as also P_1 and P_2 from the equations (5) and substitute in (6), these equations reduce to an identity. The above statement is also true geometrically, since the plane must be tangent to both curves at the point $u =$ const. $v =$ const. We may say then

To all points of a two-dimensional surface in M_5 belonging to a nullsystem there corresponds in M_3 all the surface-elements of a surface.

The converse is also true and may easily be proved.

A two-dimensional surface S_2 in M_5 is said to belong to an asymptotic complex

$$\left.\begin{aligned} dx_5 - x_1dx_2 + x_2dx_1 - x_3dx_4 + x_4dx_3 &= 0,\\ dx_1dx_2 + dx_3dx_4 &= 0,\end{aligned}\right\} \tag{7}$$

whenever the coordinate system (u), (v) belongs to the complex. The functions ϕ_i, $(i = 1, 2, 3, 4, 5)$ must now satisfy the following conditions:

$$\left.\begin{aligned} \frac{\partial\phi_5}{\partial u} = \phi_1\frac{\partial\phi_2}{\partial u} - \phi_2\frac{\partial\phi_1}{\partial u} + \phi_3\frac{\partial\phi_4}{\partial u} - \phi_4\frac{\partial\phi_3}{\partial u},\\ \frac{\partial\phi_5}{\partial v} = \phi_1\frac{\partial\phi_2}{\partial v} - \phi_2\frac{\partial\phi_1}{\partial v} + \phi_3\frac{\partial\phi_4}{\partial v} - \phi_4\frac{\partial\phi_3}{\partial v},\end{aligned}\right\} \tag{8 (a)}$$

$$\frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u} + \frac{\partial\phi_3}{\partial u}\frac{\partial\phi_4}{\partial u} = 0, \quad \frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial v} = 0. \tag{8 (b)}$$

Transforming into M_3 we obtain a surface on which (u) and (v) are asymptotic curves. In fact, introducing the coordinates X_1, X_2, X_3, P_1 and P_2 in 8 (a) and 8 (b) we obtain

$$\left.\begin{aligned} \frac{\partial X_3}{\partial u} = P_1\frac{\partial X_1}{\partial u} + P_2\frac{\partial X_2}{\partial u}, \quad \frac{\partial X_3}{\partial v} = P_1\frac{\partial X_1}{\partial v} + P_2\frac{\partial X_2}{\partial v},\\ \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u} + \frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u} = 0, \quad \frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v} + \frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v} = 0,\end{aligned}\right\} \tag{9}$$

of these equations the first two expresses the fact that the plane of the surface elements is tangent to the surface, while the two last are the conditions that this

plane shall osculate the curves (u) and (v), that is (u) and (v) are asymptotic lines.

The problem to find all the surfaces belonging to an asymptotic complex is thus seen to be reduced to the problem of finding all the surfaces in ordinary space on which (u) and (v) are asymptotic lines. For the solution of this problem the reader is referred to Vol. IV of Darboux's "Théorie des Surface," page 20. (Lelieuvre's formulae).

Conversely, the problem to find surfaces on which (u) and (v) are asymptotic lines may from the standpoint of asymptotic complexes be considered equivalent to the problem of finding all the two-dimensional surfaces belonging to such a complex. Treating the problem from this point of view we proceed as follows:*

The function ϕ_i, $(i = 1, 2, 3, 4, 5)$ in addition to satisfying the conditions 8 (a) and 8 (b) must, as we have seen, as a consequence also satisfy the equation

$$\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial u}-\frac{\partial\phi_2}{\partial v}\frac{\partial\phi_1}{\partial u}+\frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial u}-\frac{\partial\phi_4}{\partial v}\frac{\partial\phi_3}{\partial u}=0,$$

which by 8 (b) reduces to the form

$$\left(\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial v}+\frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial u}\right)\left(\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_3}{\partial u}-\frac{\partial\phi_1}{\partial u}\frac{\partial\phi_3}{\partial v}\right)=0.$$

If the second factor vanishes we obtain in M_5 a curve instead of a surface contrary to hypothesis; we may therefore exclude this case and put

$$\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial v}+\frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial u}=0. \tag{10}$$

From 8 (b) and (10) we obtain

$$\left.\begin{aligned}\frac{\partial\phi_1}{\partial u}&=-\frac{\dfrac{\partial\phi_3}{\partial u}}{\dfrac{\partial\phi_2}{\partial u}}\cdot\frac{\partial\phi_4}{\partial u}\equiv-R\frac{\partial\phi_4}{\partial u_4},\\ \frac{\partial\phi_1}{\partial v}&=-\frac{\dfrac{\partial\phi_3}{\partial v}}{\dfrac{\partial\phi_2}{\partial v}}\cdot\frac{\partial\phi_4}{\partial v}=\quad R\frac{\partial\phi_4}{\partial v}.\end{aligned}\right\} \tag{11}$$

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* **This method was developed by the writer at a time when he had no access to any literature on the subject of asymptotic lines; it is therefore strictly original, and presents the problem of Lelieuvre from an entirely new point of view.**

From (10) and (11) we obtain, since $d\phi_1$ and $d\phi_3$ must be exact differentials, the following differential equation which must be satisfied by the functions ϕ_2 and ϕ_4

$$\frac{\partial^2\theta}{\partial u\,\partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R\,.\,\frac{\partial\theta}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R\,.\,\frac{\partial\phi_2}{\partial v} = 0\,. \tag{12}$$

If two particular solutions ϕ_2 and ϕ_4 can be found the determination of ϕ_1, ϕ_3 and ϕ_5 from (11), (10) and (8) will involve quadratures only. *The problem to find all surfaces on which* (u) *and* (v) *are asymptotic lines is thus reduced to the integration of the differential equation* (18) *which is one of equal invariants and may be reduced to the form*

$$\frac{\partial^2\theta}{\partial u\,\partial v} = h_1\theta\,; \tag{12'}$$

hence the

THEOREM.—*If a surface $x_i = \phi_i(u, v)$ in M_5 belongs to an asymptotic complex the coordinates x_2 and x_4 must satisfy a differential equation of the form*

$$\frac{\partial^2\theta}{\partial u\,\partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R\,.\,\frac{\partial\theta}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R\,\frac{\partial\theta}{\partial v} = 0\,.$$

It is interesting to note that the determination of all surfaces belonging to an asymptotic complex is of the same nature as the problem of infinitesimal deformation of surfaces in ordinary space (see Darboux, "Théorie des Surfaces," Vol. IV, Ch. II).

We shall apply the above method to a few examples:

1°. Let the surface in M_5 be given in the form

$$x_1 = \lambda_1(u) + \mu_1(v)\,,\quad x_2 = \lambda_2(u) + \mu_2(v) = c\,(u - v)\,,$$
$$x_3 = \lambda_3(u) + \mu_3(v)\,,\quad x_4 = \lambda_4(u) + \mu_4(v)\,,\quad x_5 = \lambda_5(u) + \mu_5(v)\,;$$

which is a two-dimensional translation surface.

The curves (u) and (v) are two sets of parallel curves. The condition (10) becomes, on substituting,

$$\mu_3' = \lambda_3'\,,$$

which can only be possible if $\mu_3 = c_1 v$ and $\lambda_3 = c_1 u$, so that $x_3 = c_1(u + v)$. From 8 (b) we derive

$$\lambda_4' = -\frac{c}{c_1}\lambda_1'\,,\quad \mu_4' = \frac{c}{c_1}\mu_1'\,,$$

hence, we must put $x_4 = -\frac{c}{c_1}(\lambda_1 - \mu_1)$. x_5 may now be calculated from 8 (a). We find

$$x_5 = 4c\int \lambda_1 du - 2cu\lambda_1 - 4c\int \mu_1 dv + 2cv\mu_1 .$$

We now put

$$4c\int \lambda_1 du = F, \quad -4c\int \mu_1 dv = F_1,$$

and also $c_1 = \frac{1}{4}$, $c = \frac{1}{2k}$ and our surface has now the form

$$\begin{aligned} x_1 &= \frac{2}{k}(F' - F_1'), \\ x_2 &= \frac{u-v}{2k}, \\ x_3 &= \frac{u+v}{4}, \\ x_4 &= -F' - F_1', \\ x_5 &= F + F_1 - \frac{u}{2}F' - \frac{u}{2}F_1'; \end{aligned}$$

corresponding to this surface, we have in M_3

$$\begin{aligned} X_1 &= \frac{u-v}{2k}, \\ X_2 &= -F' - F_1', \\ X_3 &= F + F_1 - \frac{u+v}{2}F' - \frac{u+v}{2}F_1', \end{aligned}$$

a family of surfaces that has been obtained by Darboux (Leçons, Vol. I, p. 141) by an entirely different method; in fact, we only need to change the coordinate-system by means of the transformation

$$\frac{u-v}{2k} = \alpha, \quad \frac{u+v}{2} = \beta,$$

in order to get the identical form given by Darboux.

2°. Let $\phi_3 = uv$. We find from (10) $\phi_2 = \phi_2\left(\frac{u}{v}\right)$.

Substituting in (11) we get

$$\frac{\partial\phi_1}{\partial u} = \frac{-v^2\frac{\partial\phi_4}{\partial u}}{\phi_2'}, \quad \frac{\partial\phi_1}{\partial v} = \frac{v^2\frac{\partial\phi_4}{\partial v}}{\phi_2'}, \quad R = \frac{v^2}{\phi_2'}.$$

The differential equation (12) becomes

$$\frac{\partial^2\theta}{\partial u\,\partial v} + A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v} = 0, \tag{13}$$

where

$$A = \frac{2v\phi_2' + u\phi_2''}{2v^2\phi_2'}, \quad B = \frac{-\phi_2''}{2v\phi_2'}.$$

We may now make the special hypothesis that the invariant

$$k = \frac{\partial B}{\partial v} + AB = 0,$$

from which we obtain on integrating

$$\phi_2 = c_1\frac{v}{u} + c_2\frac{u}{v} + c_3.$$

Substituting this value in A and B and integrating the differential equation (13) we get

$$\theta = \phi_4 = e^{-\int B du}\left(\sigma(v) + \int\rho(u)\,e^{\int B du - \int A dv}\,du\right).$$

ϕ_1 and ϕ_5 may now be calculated from the equations (11) and (8). That ϕ_2 also satisfies (13) may be verified by substituting in (13) for θ the function

$$c_1\frac{v}{u} + c_2\frac{u}{v} + c_3.$$

From equation (11) we have

$$\frac{\partial\phi_2}{\partial u} = \frac{1}{R}\cdot\frac{\partial\phi_3}{\partial u}, \quad \frac{\partial\phi_2}{\partial v} = -\frac{1}{R}\frac{\partial\phi_3}{\partial v},$$

$$\frac{\partial\phi_4}{\partial u} = -\frac{1}{R}\frac{\partial\phi_1}{\partial u}, \quad \frac{\partial\phi_4}{\partial v} = \frac{1}{R}\frac{\partial\phi_1}{\partial v},$$

from which we easily derive the following differential equation similar to the equation (12)

$$\frac{\partial^2\theta}{\partial u\,\partial v} - \tfrac{1}{2}\frac{\partial}{\partial v}\log R\,\frac{\partial\theta}{\partial u} - \tfrac{1}{2}\frac{\partial}{\partial u}\log R\,\frac{\partial\theta}{\partial v} = 0, \tag{14}$$

of which ϕ_1 and ϕ_3 are particular solutions; if two such solutions can be formed ϕ_2 and ϕ_4 may be obtained from equations (11) and ϕ_5 from (8). Since it is an equation with equal invariants it may be put in the form

$$\frac{\partial^2\bar\theta}{\partial u\,\partial v} = h_2\bar\theta, \tag{14'}$$

where

$$h_2 = -\tfrac{1}{2}\frac{\partial^2 \log R}{\partial u\,\partial v} + \tfrac{1}{4}\frac{\partial}{\partial u}\log R\,.\,\frac{\partial}{\partial u}\log R.$$

The particular solutions of (14′) are

$$\bar\theta_1 = \sqrt{R}, \quad \bar\theta_2 = \phi_1\sqrt{R}, \quad \bar\theta_3 = \phi_3\sqrt{R}.$$

If then three particular solutions of (14′) *can be found, we know how to obtain a surface belonging to the asymptotic complex.*

Example. Let $h_2 = 0$; we have then

$$\bar\theta = \rho(u) + \sigma(v);$$

we may therefore put

$$\bar\theta_1 = \rho(u) + \sigma(v), \quad \bar\theta_2 = \phi_1(\rho+\sigma) = (\rho_1+\sigma_1)(\rho+\sigma),$$
$$\bar\theta_3 = \phi_3(\rho+\sigma) = (\rho_3+\sigma_3)(\rho+\sigma),$$

so that we get

$$\phi_1 = \rho_1 + \sigma_1, \quad \phi_3 = \rho_3 + \sigma_3.$$

We also find

$$h_1 = \frac{\rho'\sigma'}{(\rho+\sigma)^2}.$$

The equation (12′) therefore becomes

$$\frac{\partial^2\bar\theta}{\partial u\,\partial v} = \frac{\rho'\sigma'}{(\rho+\sigma)^2}\bar\theta.$$

which may be integrated by Laplace's method. We find, taking two particular solutions,

$$\bar{\theta}_2 = -2\frac{\rho_2+\sigma_2}{\rho+\sigma} + \frac{\rho_2'}{\rho'} + \frac{\sigma_2'}{\sigma'},$$

$$\bar{\theta}_4 = -2\frac{\rho_4+\sigma_4}{\rho+\sigma} + \frac{\rho_4'}{\rho'} + \frac{\sigma_4'}{\sigma'}.$$

but $\bar{\theta}_2 = \frac{\phi_2}{\sqrt{R}}$ and $\bar{\theta}_4 = \frac{\phi_4}{\sqrt{R}}$, hence we get

$$\phi_2 = -2(\rho_2+\sigma_2) + \left(\frac{\rho_2'}{\rho'} + \frac{\sigma_2'}{\sigma'}\right)(\rho+\sigma),$$

$$\phi_4 = -2(\rho_4+\sigma_4) + \left(\frac{\rho_4'}{\rho'} + \frac{\sigma_4'}{\sigma'}\right)(\rho+\sigma).$$

ϕ_5 may now be obtained from (8) without difficulty, since we know that $d\phi_5$ is a perfect differential. We shall not reproduce the calculations here.

We may state the chief result of the preceding development thus:

There exists a one-to-one correspondence between all the surfaces of three-dimensional space and all the two-dimensional surfaces of five-dimensional space belonging to an asymptotic complex.

Given a surface in M_3 referred to its asymptotic curves, three particular solutions of (14′) are known and the corresponding surface in M_5 may be found. The geometrical meaning of R is obvious,*

$$R^2 = s^2 - rt,$$

so that if K denotes the total curvature of the given surface, we have

$$K = \frac{-R^2}{(1+P_1^2+P_2^2)^2}.$$

As an example, we may take a sphere $X_1^2+X_2^2+X_3^2=1$ referred to its rectilinear generators which we know are asymptotic lines. We have then

$$X_1 = \frac{1-\alpha\beta}{\alpha-\beta}, \quad X_2 = i\,\frac{1+\alpha\beta}{\alpha-\beta}, \quad X_3 = \frac{\alpha+\beta}{\alpha-\beta}.$$

* See Darboux, Vol. IV, p. 21, where the quantity λ corresponds to the reciprocal of R.

Calculating P_1 and P_2, we find

$$P_1 = \frac{\alpha\beta - 1}{\alpha + \beta}, \quad P_2 = -i\,\frac{\alpha\beta + 1}{\alpha + \beta}.$$

The corresponding surface in M_5 is

$$x_1 = \tfrac{1}{2}\,\frac{\alpha\beta - 1}{\alpha + \beta}, \quad x_2 = \frac{1 - \alpha\beta}{\alpha - \beta}, \quad x_3 = -\frac{i}{2}\,\frac{1 + \alpha\beta}{\alpha + \beta},$$
$$x_4 = i\,\frac{1 + \alpha\beta}{\alpha - \beta}, \quad x_5 = \frac{\alpha^2 + \beta_2}{\alpha^2 - \beta^2}.$$

Eliminating α and β, we have the surface

$$\left.\begin{aligned} (4x_1^2 + 4x_3^2 + 1)(1 - x_2^2 - x_4^2) &= 1, \\ x_2^2 + x_4^2 + (x_5 + x_1x_2 + x_3x_4)^2 &= 1, \\ 2x_1x_5 + 2x_1^2x_2 + 2x_3^2x_2 + x_2 &= 0, \end{aligned}\right\} \tag{15}$$

which satisfies the differential equations

$$\begin{aligned} dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 &= 0, \\ dx_1dx_2 + dx_3dx_4 &= 0. \end{aligned}$$

The lines (α) and (β) are curves corresponding to the rectilinear generators of the sphere; the surface may be considered as the envelope of ∞^2 straight lines belonging to the asymptotic complex.

Given in M_3 a ruled surface

$$\left.\begin{aligned} X_1 &= \phi_2 + \psi_2 v, \\ X_2 &= \phi_4 + \psi_4 v, \\ X_3 &= \phi + \xi v, \end{aligned}\right\} \tag{16}$$

to it there will correspond in M_5 a surface on which the lines (u) belong to an asymptotic complex while the lines (v) in general will be lines of the nullsystem. When will this surface be a ruled surface? Calculating P_1 and P_2 we find

$$P_1 = \frac{\phi'\psi_4 - \xi\phi_4' + (\psi_4\xi' - \psi_4\xi)\,v}{\phi_2'\psi_4 - \phi_4'\psi_2 + (\psi_2'\psi_4 - \psi_4\psi_2)\,v},$$
$$P_2 = \frac{-\phi'\psi_2' + \xi\phi_2' + (\psi_2'\xi - \psi_2'\xi')\,v}{\phi_2'\psi_4 - \phi_4'\psi_2 + (\psi_2'\psi_4 - \psi_4'\psi_2)\,v},$$

and the surface in M_5 may be written

$$\left.\begin{aligned} &x_1 = \frac{P_1}{2}, \quad x_2 = \phi_2 + \psi_2 v, \quad x_3 = \frac{P_2}{2}, \quad x_4 = \phi_4 + \psi_4 v, \\ &x_5 = \phi = \xi v - \frac{P_1}{2}(\phi_2 + \psi_2 v) - \frac{P_2}{2}(\phi_4 + \psi_4 v), \end{aligned}\right\} \tag{16'}$$

which will be a ruled surface whenever the coefficient of v^2 on the right hand side of the last equation vanishes, that is, whenever $\psi_2'\psi_4 - \psi_4'\psi_2 = 0$, or $\psi_2 = c\psi_4$. Interpreted geometrically this means that the rectilinear generators are always parallel to a fixed plane which is perpendicular to the X_1X_2-plane; that is, we have all the ruled surfaces with a plane director.*

If we choose the functions ψ_4 and ξ in such a way that $c^2\psi_4^2 + \psi_4^2 + \xi^2 = 1$, v will denote the distance of any point on the rectilinear generator from the point where it intersects the fixed curve $v = 0$ and u will denote the angle which this line makes with its projection on the X_1X_2-plane. The surface then takes the form

$$\left.\begin{aligned} X_1 &= \phi_2 + \frac{c}{\sqrt{c^2+1}} \cos u \,.\, v, \\ X_2 &= \phi_4 + \frac{\cos u}{\sqrt{c^2+1}} \,.\, v, \\ X_3 &= \phi \; + \sin u \,.\, v. \end{aligned}\right\} \tag{17}$$

If the director-plane is transformed into the X_2X_3-plane, c will be zero and the above equations become

$$X_1 = \phi_2, \quad X_2 = \phi_4 + \cos u \,.\, v, \quad X_3 = \phi + \sin u \,.\, v. \tag{17'}$$

As is well known, to this class of surfaces belong the screw-surfaces and conoids.† If the surface is a developable surface, it is cylindrical. If the rectilinear generators belong to the nullsystem

$$X_1 dX_2 - X_2 dX_1 + k dX_3 = 0,$$

* These surfaces have been studied by Catalan, J. É.c. Polyt. 17 (1843), p. 121. See also Encyclopaedie der Mathematischen Wissenschaften, Band III$_3$, p. 271.

† Thus, if in (17′), $\phi_2 = au$, $\phi_4 = 0$, $\phi = 0$, we get the skew helicoid with plane director.

it is a developable surface and hence cylindrical. The proof of these statements we leave to the reader.

Conversely, *to every ruled surface in M_5 whose rectilinear generators (u) belong to an asymptotic complex and whose coordinate lines (v) belong to the nullsystem, there corresponds in M_3 a ruled surface with a plane-director perpendicular to the X_1X_2-plane.*

Let the surface be represented by the equations

$$x_i = \lambda_i(u) + \rho_i(u)\,v. \qquad i = 1,\ 2,\ 3,\ 4,\ 5 \tag{18}$$

The following conditions must be satisfied:

$$\left.\begin{aligned}
&\rho_5 = \rho_2\lambda_1 - \rho_1\lambda_2 + \rho_4\lambda_3 - \rho_3\lambda_4,\\
&\lambda_5' = \lambda_1\lambda_2' - \lambda_2\lambda_1' + \lambda_3\lambda_4' - \lambda_4\lambda_3',\\
&\rho_1\rho_2 + \rho_3\rho_4 = 0,\\
&\rho_2\rho_1' - \rho_1\rho_2' + \rho_3'\rho_4 - \rho_3\rho_4' = 0,\\
&\rho_5' + \rho_2\lambda_1' + \rho_1'\lambda_2 - \rho_1\lambda_2' - \lambda_1\rho_2' + \lambda_4\rho_3' + \lambda_3'\rho_4 - \lambda_4'\rho_3 - \lambda_3\rho_4' = 0.
\end{aligned}\right\} \tag{19}$$

These conditions may be further simplified. We differentiate the first and substitute the value of ρ_5' thus found in the last equation which reduces to

$$\lambda_2'\rho_1 - \lambda_1'\rho_2 + \lambda_4'\rho_3 - \lambda_3'\rho_4 = 0. \tag{20}$$

Differentiating the third and adding to the fourth, we obtain

$$\rho_2\rho_1' + \rho_4\rho_3' = 0. \tag{21}$$

From the third of equations (19) and from (21) we obtain

$$\frac{\rho_2}{\rho_4} = -\frac{\rho_3}{\rho_1} = -\frac{\rho_3'}{\rho_1'},$$

which is satisfied by putting

$$\rho_3 = -c\rho_1, \quad \rho_2 = c\rho_4.$$

Equation (20) now becomes

$$\lambda_2' - c\lambda_4' = \frac{\rho_4}{\rho_1}(c\lambda_1' + \lambda_3'),$$

which determines λ_3; ρ_5 and λ_5 may now be obtained from the first of equations (19). The surface in M_3 is

$$\begin{aligned} X_1 &= \lambda_2 + c\rho_4 v, \\ X_2 &= \lambda_4 + \rho_4 v, \\ X_3 &= \lambda_5 + \lambda_1\lambda_2 + \lambda_3\lambda_4 + 2\rho_4(c\lambda_1 + \lambda_3)v, \end{aligned}$$

which is a ruled surface whose plane director is perpendicular to the X_1X_2-plane.

We shall now prove the following

THEOREM.— *The second set of asymptotic lines of all ruled surfaces with a plane director may be obtained by quadratures.*

Choose for X_1X_2-plane any plane perpendicular to the plane director; the equations at the surface will be of the form

$$X_1 = \phi_2 + c\psi_4 v, \quad X_2 = \phi_4 + \psi_4 v, \quad X_3 = \phi + \xi v. \tag{22}$$

Calculating P_1 and P_2 we find,

$$\left.\begin{aligned} P_1 &= \frac{\phi'\psi_4 - \xi\phi_4' + \psi_4(\xi' - \xi)v}{\psi_4(\phi_2' - c\phi_4')} = \lambda_1 + \rho_1 v, \\ P_2 &= \frac{-c\phi'\psi_4 + \xi\phi_2' - c\psi_4(\xi' - \xi)v}{\psi_4(\phi_2' - c\phi_4')} = \lambda_3 - c\rho_1 v. \end{aligned}\right\} \tag{22'}$$

The differential equation of the asymptotic lines is

$$dX_1dP_1 + dX_2dP_2 = 0.$$

Substituting in this equation the values of dX_1, dX_2, dP_1, dP_2 obtained by differentiating equations (22) and (22′) we get an equation of the form

$$\frac{dv}{du} = A + Bv, \tag{23}$$

which is integrable by quadratures. Q. E. D.

From this theorem it follows that the second set of lines belonging to an asymptotic complex on the given ruled surface in M_5 of the form (13), the ρ's and λ's satisfying the conditions (19), can be obtained by quadratures. The differential equation which determines this system is

$$\frac{dv}{du} = \frac{\lambda_1'\lambda_2' + \lambda_3'\lambda_4' + \dfrac{\rho_1'\rho_4 + \rho_4'\rho_1}{\rho_1}(\lambda_3' + c\lambda_1')cv}{-2\rho_4(\lambda_3' + c\lambda_1')};$$

which is of the form (23).

III.

A point-transformation in M_5 will as a rule transform the ∞^7 lines of the nullsystem into curves; if these curves are to be straight lines, the transformation must be projective; in fact, a line is defined by four linear spaces and a transformation that transforms a linear space into a linear space must necessarily be projective. There are in M_5 ∞^{35} such transformations defining the projective group of that space. Among all these transformations we shall consider those that transform any line of the nullsystem into some other line of the same system. These transformations define a group which we shall call with Lie *the projective group of the nullsystem.** This group is made up of the following 21 separate transformations:

$$\left.\begin{array}{l}
x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}, \quad x_3\frac{\partial f}{\partial x_2} + x_1\frac{\partial f}{\partial x_4}, \quad \frac{\partial f}{\partial x_5}, \quad x_2\frac{\partial f}{\partial x_1}, \quad x_1\frac{\partial f}{\partial x_2}, \quad x_4\frac{\partial f}{\partial x_3}, \quad x_3\frac{\partial f}{\partial x_4}, \\
x_1\frac{\partial f}{\partial x_3} - x_4\frac{\partial f}{\partial x_4}, \quad x_4\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_3}, \quad \frac{\partial f}{\partial x_2} - x_1\frac{\partial f}{\partial x_5}, \quad \frac{\partial f}{\partial x_4} - x_3\frac{\partial f}{\partial x_5}, \\
x_4\frac{\partial f}{\partial x_3} - x_3\frac{\partial f}{\partial x_4}, \quad \frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_5}, \quad \frac{\partial f}{\partial x_3} + x_4\frac{\partial f}{\partial x_5}, \quad x_2\frac{\partial f}{\partial x_4} - x_3\frac{\partial f}{\partial x_1}, \\
x_5\frac{\partial f}{\partial x_2} - x_2\sum_1^5 x_i\frac{\partial f}{\partial x_i}, \quad x_5\frac{\partial f}{\partial x_1} + x_1\sum_1^5 x_i\frac{\partial f}{\partial x_i}, \quad x_5\frac{\partial f}{\partial x_4} - x_4\sum_1^5 x_i\frac{\partial f}{\partial x_i}, \\
x_5\frac{\partial f}{\partial x_3} + x_3\sum_1^5 x_i\frac{\partial f}{\partial x_i}, \quad x_5\frac{\partial f}{\partial x_5} + \sum_1^5 x_i\frac{\partial f}{\partial x_i}, \quad x_5\sum x_i\frac{\partial f}{\partial x_i}.
\end{array}\right\} \quad (1)$$

But to any point-transformation in M_5 leaving the nullsystem invariant there corresponds in M_3 a contact-transformation. In fact, from the invariance of the Pfaffian equation

$$dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 = 0,$$

there follows by virtue of the transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_5 + x_1x_2 + x_3x_4 = X_3 \qquad (2)$$

also the invariance of

$$dX_3 - P_1dX_1 - P_2dX_2 = 0,$$

which means that the transformation in M_3 is a contact-transformation. The group of contact-transformations may be found by substituting in (1) the new variable from (2). We find then the group

* See Lie, Theorie der Transformationsgruppen, Ab. II, p. 521.

$$\left.\begin{aligned}
&P_1\frac{\partial f}{\partial P_1}-X_1\frac{\partial f}{\partial X_1},\quad X_2\frac{\partial f}{\partial X_2}-P_2\frac{\partial f}{\partial P_2},\quad X_1\frac{\partial f}{\partial X_3}+\frac{\partial f}{\partial P_1},\quad X_2\frac{\partial f}{\partial X_3}+\frac{\partial f}{\partial P_2},\\
&P_1\frac{\partial f}{\partial P_2}-X_2\frac{\partial f}{\partial X_1},\quad X_1\frac{\partial f}{\partial X_2}-P_2\frac{\partial f}{\partial P_1},\quad \frac{\partial f}{\partial X_1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \frac{\partial f}{\partial X_2},\quad \frac{\partial f}{\partial X_3},\quad P_2^2\frac{\partial f}{\partial X_3}+2P_2\frac{\partial f}{\partial X_2}\\
&2X_3\frac{\partial f}{\partial X_3}+P_1\frac{\partial f}{\partial P_1}+P_2\frac{\partial f}{\partial P_2}+X_1\frac{\partial f}{\partial X_1}+X_2\frac{\partial f}{\partial X_2},\\
&\qquad\qquad P_1^2\frac{\partial f}{\partial X_3}+2P_1\frac{\partial f}{\partial X_1},\quad X_2\frac{\partial f}{\partial P_1}+X_1\frac{\partial f}{\partial P_2}+X_1X_2\frac{\partial f}{\partial X_3},\\
&X_2^2\frac{\partial f}{\partial X_3}+2X_2\frac{\partial f}{\partial P_2},\quad P_2\frac{\partial f}{\partial X_1}+P_1\frac{\partial f}{\partial X_2}+P_1P_2\frac{\partial f}{\partial X_3},\quad X_1^2\frac{\partial f}{\partial X_3}+2X_1\frac{\partial f}{\partial P_1},\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\frac{\partial f}{\partial P_1}+\tfrac{1}{2}X_1\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum_1^3 X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\frac{\partial f}{\partial P_2}+\tfrac{1}{2}X_2\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum_1^3 X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\Big(\frac{\partial f}{\partial X_1}+P_1\frac{\partial f}{\partial X_3}\Big)-\frac{P_1}{2}\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\Big(\frac{\partial f}{\partial X_1}+P_2\frac{\partial f}{\partial X_3}\Big)-\frac{P_2}{2}\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]^2\frac{\partial f}{\partial X_3}-[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big].
\end{aligned}\right\}\quad(3)$$

This group transforms the ∞^5 parabolae and their surface-elements into themselves. But the parabolae are the integral curves of the differential equations

$$\frac{d^3X_3}{dX_1^3}=0,\quad \frac{d^2X_2}{dX_1^2}=0.$$

Hence we have the

THEOREM.—*There exists in ordinary space* ∞^{21} *contact-transformations which leave the differential equations*

$$\frac{d^3X_3}{dX_1^3}=0,\quad \frac{d^2X_2}{dX_1^2}=0 \tag{4}$$

invariant. Moreover, there are 21 *infinitesimal contact-transformations leaving these equations invariant.*

Of all the projective transformations that leave the nullsystem invariant, the Euclidian motion presents the greatest interest. That such a motion exists is already evident from the definition of a nullsystem according to which the line-elements of the system move perpendicular to the direction of the n-dimensional screw-motion defined by the equations

$$\frac{\delta x_1}{\delta t}=x_2,\quad \frac{\delta x_2}{\delta t}=-x_1,\ \ldots\ldots,\ \frac{\delta x_{u-4}}{\delta t}=x_{u-3},\quad \frac{\delta x_{u-3}}{\delta t}=-x_{u-4},$$
$$\frac{\delta x_3}{\delta t}=x_4,\quad \frac{\delta x_4}{\delta t}=-x_3,\ \ldots\ldots,\ \frac{\delta x_{u-2}}{\delta t}=x_{u-1},\quad \frac{\delta x_{u-1}}{\delta t}=-x_{u-2},$$
$$\frac{\delta x_u}{\delta t}=c_u.$$

The system must therefore remain invariant during this motion.

Are there other Euclidian motions of the same nature? To answer this question we shall first consider the case $n=5$ and construct the group of Euclidian motions, leaving the nullsystem of M_5 invariant. A Euclidian transformation group is defined by the system of equations

$$\left.\begin{aligned}
\frac{\delta x_1}{\delta t}&=\phantom{-\alpha_{12}x_1+{}}\alpha_{12}x_2+\alpha_{13}x_3+\alpha_{14}x_4+\alpha_{15}x_5+c_1,\\
\frac{\delta x_2}{\delta t}&=-\alpha_{12}x_1+\alpha_{23}x_3+\alpha_{24}x_4+\alpha_{25}x_5+c_2,\\
\frac{\delta x_3}{\delta t}&=-\alpha_{13}x_1-\alpha_{23}x_2+\alpha_{34}x_4+\alpha_{35}x_5+c_3,\\
\frac{\delta x_4}{\delta t}&=-\alpha_{14}x_1-\alpha_{24}x_2-\alpha_{34}x_3+\alpha_{45}x_5+c_4,\\
\frac{\delta x_5}{\delta t}&=-\alpha_{15}x_1-\alpha_{25}x_2-\alpha_{35}x_3-\alpha_{45}x_4+c_5.
\end{aligned}\right\} \tag{5}$$

Since this transformation is to leave the nullsystem invariant, we must have

$$\delta\left(dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4\right) \equiv 0,$$

or, what is the same thing,

$$d\delta x_5 + \delta x_2 dx_1 + x_2 d\delta x_1 \ldots . \equiv 0.$$

Substituting in this equation the values of δx_i taken from (5) and equating to zero the coefficients of $x_i dx_k$ and dx_i, $(k,\ i = 1,\ 2,\ 3,\ 4,\ 5)$, we obtain

$$\begin{gathered}\alpha_{15} = \alpha_{25} = \alpha_{35} = \alpha_{45} = c_1 = c_2 = c_3 = c_4 = 0,\\ \alpha_{13} - \alpha_{24} = 0, \quad \alpha_{14} + \alpha_{23} = 0,\end{gathered}$$

so that the system (5) may now be written

$$\left.\begin{aligned}\frac{\delta x_1}{\delta t} &= \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4,\\ \frac{\delta x_2}{\delta t} &= -\alpha_{12}x_1 - \alpha_{14}x_3 + \alpha_{13}x_4,\\ \frac{\delta x_3}{\delta t} &= -\alpha_{13}x_1 + \alpha_{14}x_2 + \alpha_{34}x_4,\\ \frac{\delta x_4}{\delta t} &= -\alpha_{14}x_1 - \alpha_{14}x_2 + \alpha_{34}x_3,\\ \frac{\delta x_5}{\delta t} &= c_5,\end{aligned}\right\} \qquad (5')$$

which shows that the group is composed of the following independent infinitesimal transformations

$$\begin{gathered}x_2\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial x_2}, \quad x_4\frac{\partial f}{\partial x_3} - x_3\frac{\partial f}{\partial x_4}, \quad \frac{\partial f}{\partial x_5},\\ x_3\frac{\partial f}{\partial x_1} + x_4\frac{\partial f}{\partial x_2} - x_1\frac{\partial f}{\partial x_3} - x_2\frac{\partial f}{\partial x_4}, \quad x_4\frac{\partial f}{\partial x_1} - x_3\frac{\partial f}{\partial x_2} + x_2\frac{\partial f}{\partial x_3} - x_1\frac{\partial f}{\partial x_4},\end{gathered} \qquad (6)$$

that is to say *four rotations* and a translation along the x_5 — axis. It is not at all difficult to extend this method to any odd number of dimensions. We obtain

after some lengthy calculations the following system of equations:

$$\frac{\delta x_1}{\delta t} = \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4 + \alpha_{15}x_5 + \alpha_{16}x_6 + \cdots + \alpha_{1n-1}x_{n-1},$$

$$\frac{\delta x_2}{\delta t} = -\alpha_{12}x_1 - \alpha_{14}x_3 + \alpha_{13}x_4 + \alpha_{16}x_5 + \alpha_{15}x_6 + \cdots + \alpha_{1n-2}x_{n-1},$$

$$\frac{\delta x_3}{\delta t} = -\alpha_{12}x_1 + \alpha_{14}x_2 + \alpha_{34}x_4 + \alpha_{35}x_5 + \alpha_{36}x_6 + \cdots + \alpha_{3n-1}x_{n-1},$$

$$\frac{\delta x_4}{\delta t} = -\alpha_{14}x_1 - \alpha_{13}x_2 - \alpha_{34}x_3 - \alpha_{36}x_5 + \alpha_{35}x_6 - \cdots + \alpha_{3n-2}x_{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\delta x_{n-1}}{\delta t} = -\alpha_{1n-1}x_1 - \alpha_{1n-2}x_2 - \alpha_{3n-1}x_3 - \cdots - \alpha_{n-2,n-2}x_{n-1},$$

$$\frac{\delta x_n}{\delta t} = c_n,$$

from which we obtain a group containing $\left(\frac{n-1}{2}\right)^2$ rotations and a translation along the x_n-axis. Hence the

THEOREM.—*The most general Euclidian motion in M_5, leaving the nullsystem*

$$dx_n + \sum_{i=1}^{i=\frac{n-1}{2}} (x_{2i}dx_{2i-1} - x_{2i-1}dx_{2i}) = 0$$

invariant, is made up of $\left(\frac{n-1}{2}\right)^2$ rotations around the origin and a translation along the x_n-axis. Among these $\left(\frac{n-1}{2}\right)^2$ rotations are also included the $\frac{n-1}{2}$ rotations of the n-dimensional screw-motion.

For $n = 5$ we can easily deduce the finite equations of the transformation group (6). If we extend the method described in Lie-Scheffer's "Vorlesungen über Continuierliche Gruppen," p. 196, to the case of 5 variables, we obtain the equations

$$\begin{aligned}
\bar{x}_1 &= x_1(\cos t_1 \cos t_2 \cos t_3 + \sin t_1 \sin t_2 \sin t_3) \\
&\quad + x_2(\sin t_1 \cos t_2 \cos t_3 - \sin t_1 \sin t_2 \sin t_3) + x_3 \sin t_2 \cos t_3 + x_4 \cos t_2 \sin t_3, \\
\bar{x}_2 &= x_1(\cos t_1 \sin t_2 \sin t_3 - \sin t_1 \cos t_2 \cos t_3) \\
&\quad + x_2(\cos t_1 \cos t_2 \cos t_3 + \sin t_1 \sin t_2 \sin t_3) - x_3 \cos t_2 \sin t_3 + x_4 \sin t_2 \cos t_3, \\
\bar{x}_3 &= -x_1[\sin t_3 \cos t_2 \sin(t_1+t_4) + \sin t_2 \cos t_3 \cos(t_1+t_4)] \\
&\quad + x_2[\cos t_2 \sin t_3 \cos(t_1+t_4) - \sin t_2 \cos t_3 \sin(t_1+t_4)] \\
&\quad + x_3(\cos t_2 \cos t_3 \cos t_4 - \sin t_2 \sin t_3 \sin t_4) \\
&\quad + x_4(\cos t_2 \cos t_3 \sin t_4 + \sin t_2 \sin t_3 \cos t_4), \\
\bar{x}_4 &= -x_1[\cos t_2 \sin t_3 \cos(t_1+t_4) - \sin t_2 \cos t_3 \sin(t_1+t_4)] \\
&\quad - x_2[\cos t_3 \sin t_2 \cos(t_1+t_4) + \cos t_2 \sin t_3 \sin(t_1+t_4)] \\
&\quad - x_3(\cos t_2 \cos t_3 \sin t_4 + \sin t_2 \sin t_3 \cos t_4) \\
&\quad + x_4(\cos t_2 \cos t_3 \cos t_4 - \sin t_3 \sin t_4 \sin t_2), \\
\bar{x}_5 &= x_5 + t_5.
\end{aligned}$$

The separate transformations of the group may now be obtained by putting

$$t_2 = t_3 = t_4 = t_5 = 0\,; \quad t_1 = t_2 = t_3 = t_5 = 0\,; \quad t_1 = t_3 = t_4 = t_5 = 0\,;$$
$$t_1 = t_2 = t_4 = t_5 = 0\,; \quad t_1 = t_2 = t_3 = t_4 = 0$$

in succession; we get

$$\text{(a)}\begin{cases}\bar{x}_1 = x_1 \cos t_1 + x_2 \sin t_1,\\ \bar{x}_2 = -x_1 \sin t_1 + x_2 \cos t_1,\\ \bar{x}_3 = x_3,\\ \bar{x}_4 = x_4,\\ \bar{x}_5 = x_5,\end{cases} \qquad \text{(b)}\begin{cases}\bar{x}_1 = x_1,\\ \bar{x}_2 = x_2,\\ \bar{x}_3 = x_3 \cos t_4 + x_4 \sin t_4,\\ \bar{x}_4 = -x_3 \sin t_4 + x_4 \cos t_4,\\ \bar{x}_5 = x_5,\end{cases}$$

$$\text{(c)}\begin{cases}\bar{x}_1 = x_1 \cos t_2 + x_3 \sin t_2,\\ \bar{x}_2 = x_2 \cos t_2 + x_4 \sin t_2,\\ \bar{x}_3 = -x_1 \sin t_2 + x_3 \cos t_2,\\ \bar{x}_4 = -x_2 \sin t_2 + x_4 \cos t_2,\\ \bar{x}_5 = x_5,\end{cases} \quad \text{(d)}\begin{cases}\bar{x}_1 = x_1 \cos t_3 + x_4 \sin t_3,\\ \bar{x}_2 = x_2 \cos t_3 - x_3 \sin t_3,\\ \bar{x}_3 = x_2 \sin t_3 + x_3 \cos t_3,\\ \bar{x}_4 = -x_1 \sin t_3 + x_4 \cos t_3,\\ \bar{x}_5 = x_5,\end{cases} \quad \text{(e)}\begin{cases}\bar{x}_1 = x_1,\\ \bar{x}_2 = x_2,\\ \bar{x}_3 = x_3,\\ \bar{x}_4 = x_4,\\ \bar{x}_5 = x_5 + t_5.\end{cases}$$

Of the four rotations (a), (b), (c) and (d) the rotation (c) occupies a peculiar position in as much as *it is the only one that will make the nullsystem invariant in case it degenerates into an asymptotic complex,* that is to say, it is the only one that will render invariant the two differential equations

$$\begin{aligned}
&dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0\,,\\
&dx_1 dx_2 + dx_3 dx_4 = 0\,.
\end{aligned}$$

The group represented by

$$Uf = x_3 \frac{\partial f}{\partial x_1} + x_4 \frac{\partial f}{\partial x_2} - x_1 \frac{\partial f}{\partial x_3} - x_2 \frac{\partial f}{\partial x_3}$$

is therefore characteristic of an asymptotic complex and expresses the fact that *such a complex has only one degree of mobility*, when we except the translation along the x_5-axis which is common to all the complexes that can be formed from a given non-special nullsystem in M_5.

The following question now suggests itself: What projective transformation will transform an asymptotic complex into itself? To attack this problem directly involves rather extensive formula work; but if we remember that the lines of the complex are transformed into all the lines of M_3 and their surface elements, the question resolves itself into finding all the contact-transformations which will transform these lines into themselves. Such a transformation must evidently transform, 1°, either plane into plane and point into point, or 2°, plane into point and point into plane; that is to say, it must be either a projective transformation or a dualistic transformation. We shall consider the former first. What must be the nature of this projective transformation? According to the method by which we obtained the ∞^4 lines of M_3 (see p. 123), any straight line may be considered as a degenerate parabola lying in a plane parallel to the X_3-axis. But each parabola is the intersection of this plane with a parabolic cylinder parallel to the X_2-axis; this cylinder is tangent to the plane at infinity. The required transformation must, therefore, transform parabolic cylinders into parabolic cylinders and consequently the plane at infinity into itself. It follows, then, that the transformation must be linear; furthermore, it must transform planes parallel to the X_3-axis into planes parallel to the same axis, so that finally we arrive at the following transformation:

$$\left.\begin{aligned} \overline{X}_1 &= a_1X_1 + b_1X_2 + d_1, \\ \overline{X}_2 &= a_2X_1 + b_2X_2 + d_2, \\ \overline{X}_3 &= a_3X_1 + b_3X_2 + c_3X_3 + d_3, \end{aligned}\right\} \qquad (7)$$

$$\overline{P}_1 = \frac{b_2c_3}{a_1b_2 - a_2b_1} P_1 - \frac{a_2c_3}{a_1b_2 - a_2b_1} P_2 + \frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1},$$

$$\overline{P}_2 = \frac{-b_1c_3}{a_1b_2 - a_2b_1} P_1 + \frac{a_1c_3}{a_1b_2 - a_2b_1} P_2 + \frac{a_1b_3 - a_3b_1}{a_1b_2 - a_2b_1}.$$

The last two equations are obtained from the first three by extension* (ERWEITERUNG).

The corresponding group of infinitesimal contact-transformations contains 10 independent transformations

$$\frac{\partial f}{\partial X_1},\ \frac{\partial f}{\partial X_2},\ \frac{\partial f}{\partial X_3},\ X_1\frac{\partial f}{\partial X_1} - P_1\frac{\partial f}{\partial P_1},\ X_1\frac{\partial f}{\partial X_2} - P_2\frac{\partial f}{\partial P_1},\ X_2\frac{\partial f}{\partial X_1} - P_1\frac{\partial f}{\partial X_2},$$

$$X_2\frac{\partial f}{\partial X_2} - P_2\frac{\partial f}{\partial P_2},\ X_1\frac{\partial f}{\partial X_3} + \frac{\partial f}{\partial P_1},\ X_2\frac{\partial f}{\partial X_3} + \frac{\partial f}{\partial P_2},\ \sum_1^2\left(X_i\frac{\partial f}{\partial X_i} + P_i\frac{\partial f}{\partial P_i}\right) + 2X_3\frac{\partial f}{\partial X_3}.$$

In the space M_5 we obtain a linear projective transformation

$$\left.\begin{aligned}
2\bar{x}_1 &= \frac{2b_2c_3}{a_1b_2 - a_2b_1}x_1 - \frac{2a_2c_3}{a_1b_2 - b_1a_2}x_3 + \frac{a_3b_2 - a_2b_3}{a_1b_2 - b_1a_2},\\
\bar{x}_2 &= a_1x_2 + b_2x_4 + d,\\
2\bar{x}_3 &= -\frac{2b_1c_3}{a_1b_2 - a_2b_1}x_1 + \frac{2a_1c_3}{a_1b_2 - a_2b_1}x_3 + \frac{a_1b_3 - a_3b_1}{a_1b_2 - b_1a_2},\\
\bar{x}_4 &= a_2x_2 + b_2x_4 + d_2,\\
\bar{x}_5 &= \frac{a_3}{2}x_2 + \frac{b_3}{2}x_4 + c_3x_5 + \tfrac{1}{2}\left[\frac{d_1(a_3b_2 - a_2b_3) + d_2(a_1b_3 - a_3b_1)}{a_1b_2 - b_1a_2}\right] + d_3,
\end{aligned}\right\} \quad (8)$$

which will give rise to an infinitesimal group of 10 independent transformations

$$x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2},\ x_2\frac{\partial f}{\partial x_4} - x_3\frac{\partial f}{\partial x_1}.\ x_4\frac{\partial f}{\partial x_2} - x_1\frac{\partial f}{\partial x_4},\ x_4\frac{\partial f}{\partial x_1} - x_3\frac{\partial f}{\partial x_3},$$

$$x_4\frac{\partial f}{\partial x_5} + \frac{\partial f}{\partial x_3},\ x_2\frac{\partial f}{\partial x_5} + \frac{\partial f}{\partial x_1},\ 2x_5\frac{\partial f}{\partial x_5} + \sum_1^4 x_i\frac{\partial f}{\partial x_i},$$

$$\frac{\partial f}{\partial x_2},\ \frac{\partial f}{\partial x_4},\ \frac{\partial f}{\partial x_5}.$$

We have thus found the following

THEOREM.—*There exist in the space M_5 ∞^{10} projective transformations for which an asymptotic complex remains invariant.*

All the transformations that transform the lines of an asymptotic complex into themselves are included in these ∞^{10} projective transformations and a

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* Lie, Theorie d. Transformationsgr., Ab. II, p. 46.

dualistic transformation. This latter transformation remains to be investigated. Since it must leave invariant the equation

$$dX_3 - P_1 dX_1 - P_2 dX_2 = 0$$

and also

$$dX_1 dP_1 + dX_2 dP_2 = 0,$$

it can be no other than the classic one of Euler's, viz.:

$$\left.\begin{array}{lll} \overline{X}_1 = P_2, & \overline{X}_2 = -P_1, & \overline{X}_3 = X_3 - X_1 P_1 - X_2 P_2, \\ \overline{P}_1 = -X_2, & & \overline{P}_2 = +X_1, \end{array}\right\} \tag{9}$$

corresponding to which we have in M_5 the transformation

$$\bar{x} = 2x_3, \quad \bar{x}_4 = -2x_1, \quad 2\bar{x}_1 = -x_4, \quad 2\bar{x}_3 = +x_2, \quad \bar{x}_5 = x_5, \tag{9'}$$

which leaves invariant the asymptotic complex. All transformations leaving the asymptotic complex invariant are thus made up of a combination of this transformation and any one of the group of ∞^{10} projective transformations (8).

The relation of Euler's transformation (9) to asymptotic complexes and its effect on a⸱⸱⸱nptotic curves I shall discuss in another paper.

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On the Forms of Unicursal Quintic Curves.

By Peter Field.

Introduction.

The purpose of this paper is to study the forms of plane unicursal quintic curves. The basis of classification is the same as that used by Meyer,* but the method of obtaining the curves is entirely different. Meyer classified the curves according to the sequence of the double points and his scheme was similar to that employed by Tait † in his first paper on knots. The essential difference is that nugatory knots may appear. Suppose a curve has the six double points A, B, C, D, E, F, and suppose a variable point to describe the curve. Whenever the generating point passes through a node, indicate this fact by writing the symbol of that node. When a loop is formed near a node A the symbol would be .... AA .... for that part. Two curves which have different symbols will be regarded as distinct. A more specific classification would be one which considers the reality and positions of the inflexions, but this would lead to an almost infinite variety of forms.

In this way Meyer first wrote the 29 possible schemes for nondegenerate curves, and gave a figure for each one. He gave no equations and did not extend the method to other types of quintic curves. In his second paper he corrected some errors made in the first and showed that all the types can be obtained by quadric inversion.

Dowling ‡ devotes an article in his dissertation to the unicursal quintic curves. He derives an equation of a different form from the one that will be used in this paper. He does not give any figures. These and the references given on p. 219 of Loria's Spezielle Algebraische Kurven are the only papers known to me which deal with plane unicursal quintic curves.

In a few cases, more than one figure will be given having the same sequence

* (a) Anwendungen der Topologie auf die Gestalten der algebraischen Curven vierter und fünfter Ordnung. Muenchen (Diss.), 1878.

(b) Proceedings of the Edinburgh Royal Society, 1886. † Edinburgh Transactions, 1876–77.

‡ On the forms of plane quintic curves, Mathematical Review, Vol. 1.

of points but which nevertheless are so different in appearance as to deserve separate notice.

The following table gives Plücker's and Klein's numbers:

A is the number of real acnodes.

T the number of real double tangents with imaginary points of contact.

I the number of real inflexions.

K the number of real cusps.

n	δ	κ	τ	i
8	6	0	12	9
A	T	I		
0	0	3		
	1	1		
1	0	5		
	1	3		
	2	1		
2	0	7		
	1	5		
	2	3		
	3	1		
3	0	9		
	1	7		
	2	5		
	3	3		
	4	1		
4	1	9		
	2	7		
	3	5		
	4	3		
	5	1		
5	2	9		
	3	7		
	4	5		
	5	3		
	6	1		
6	3	9		
	4	7		
	5	5		
	6	3		
	7	1		

n	δ	κ	τ	i
7	5	1	8	7
A	T	I		
0	0	3		
	1	1		
1	0	5		
	1	3		
	2	1		
2	0	7		
	1	5		
	2	3		
	3	1		
3	1	7		
	2	5		
	3	3		
	4	1		
4	2	7		
	3	5		
	4	3		
	5	1		
5	3	7		
	4	5		
	5	3		
	6	1		

n	δ	κ	τ	i
6	4	2	5	5
K	A	T	I	
0	0	0	1	
0	1	0	3	
		1	1	
0	2	0	5	
		1	3	
		2	1	
0	3	1	5	
		2	3	
		3	1	
0	4	2	5	
		3	3	
		4	1	
2	0	0	3	
		1	1	
2	1	0	5	
		1	3	
		2	1	
2	2	1	5	
		2	3	
		3	1	
2	3	2	5	
		3	3	
		4	1	
2	4	3	5	
		4	3	
		5	1	

n	δ	κ	τ	i
5	3	3	3	3
K	A	T	I	
1	0	0	1	
1	1	0	3	
		1	1	
1	2	1	3	
		2	1	
1	3	2	3	
		3	1	
3	0	0	3	
		1	1	
3	1	1	3	
		2	1	
3	2	2	3	
		3	1	
3	3	3	3	

n	δ	κ	τ	i
4	2	4	2	1
K	A	T	I	
0	1	0	1	
0	2	1	1	
2	0	0	1	
2	1	1	1	
2	2	2	1	
4	0	1	1	
3	1	2	1	

1.—*Quintic curves with six distinct nodes.*

The equation of the fifth degree contains twenty constants so that if the six nodal points are fixed but two additional conditions may be imposed in order to fix the curve. The general equation of such a curve will now be derived and the form of the curves determined by its aid.

Let 1, 2, 3, 4, 5, 6 be the six double points; let U_0 and U_1 be two nodal cubics through these points, U_0 having a node at 1 and U_1 at 2; further let ϕ_0 and ϕ_1 be two conics, the former through 2, 3, 4, 5, 6 and the latter through 1, 3, 4, 5, 6. Then the equation

$$\phi_0 U_0 - \lambda \phi_1 U_1 = 0,$$

is the equation of any quintic curve having the given points as double points. For the curve might be defined by giving another point and the tangent at the given point. But each of the curves U_0 and U_1 contains an arbitrary constant, hence the given point may be taken as their ninth intersection and the value of λ can be taken so as to give the desired slope.

The above equations might also be considered as representing the intersections of corresponding rays of the pencils $\lambda U_0 - U_1 t = 0$ and $\phi_0 - \phi_1 t = 0$, in which t is the parameter. The points 3, 4 and 5, 6 may be conjugate imaginary and the cubics and conics are nevertheless real, but in case all six of the double points are imaginary the above form of equation is no longer applicable. In that case suppose 1 and 2, 3 and 4, 5 and 6 are conjugate imaginary points. Let α_1 be a line through 1 and 2, α_2 through 3 and 4, α_3 through 5 and 6, also let

$$\phi_1 + i\lambda_1\psi_1, \quad \phi_1 - i\lambda_1\psi_1, \quad \phi_2 + i\lambda_2\psi_2, \quad \phi_2 - i\lambda_2\psi_2, \quad \phi_3 + i\lambda_3\psi_3, \quad \phi_3 - i\lambda_3\psi_3,$$

be conics through the points

1, 3, 4, 5, 6   2, 3, 4, 5, 6   1, 2, 4, 5, 6   1, 2, 3, 5, 6   1, 2, 3, 4, 6   1, 2, 3, 4, 5

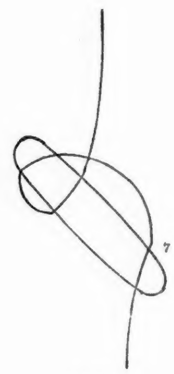
respectively. Then the equation:

$$\alpha_1(\phi_1^2 + \lambda_1^2\psi_1^2) + \varkappa_2\alpha_2(\phi_2^2 + \lambda_2^2\psi_2^2) + \varkappa_3\alpha_3(\phi_3^2 + \lambda_3^2\psi_3^2) = 0,$$

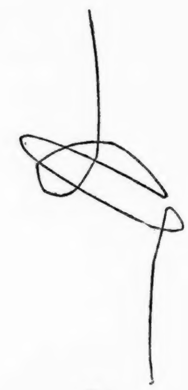
or

$$\alpha_1(\phi_1^2 + \psi_1^2) + \varkappa_2\alpha_2(\phi_2^2 + \psi_2^2) + \varkappa_3\alpha_3(\phi_3^2 + \psi_3^2) = 0,$$

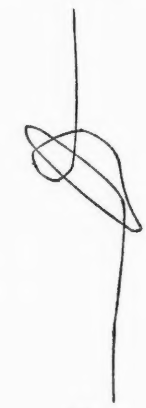
represents any quintic curve having the given imaginary double points. In



(a)

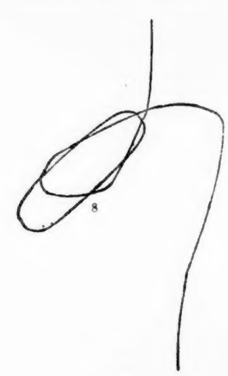


(b)



(c)

figure (a) the point 1 and any five of the intersections of ϕ_0 and U_0 may be taken as the six double points. Suppose the intersection numbered 7 is the one which is not a double point. If λ is taken very small the quintic will be very nearly of the form of U_0 and ϕ_0 except that it will not pass through the point 7. The form of the curve will be that given in figures (b) or (c) depending on the sign of λ. This shows that by taking a conic and cubic placed in the various possible relative positions and making a break as in the above figures, unicursal quintic curves result. This is the idea which has been used in constructing the curves given in this paper. By taking the conic through the node of the cubic and then proceeding as above, the forms of the curves having a triple point can be obtained. It is to be noticed that it is no restriction to always suppose that the cubic has but one infinite branch, provided the conic is not restricted.

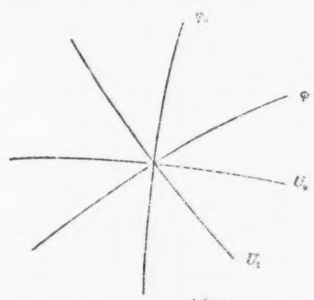


(d)

Suppose figure (d) represents $\phi_1 U_1$ and that 8 is the point which is not a node. Then, as λ grows larger, the curve changes from that given by $U_0\phi_0$

(with a break at 7) to that given by $U_1\phi_1$ (with a break at 8). In the course of this change, each double point is successively a crunode, cusp, acnode, cusp, and finally, again a crunode when ϕ, U are not divided at the nodes. No attempt has been made to trace the curves through these changes. Four of the curves given by Meyer cannot be obtained by this method unless the conic is taken as a pair of lines. This does not mean that the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$ is lacking in generality, but that the four given curves correspond to intermediate values of λ. In the case of the triple point there is no difficulty of this kind, for it can be shown by the same methods that have been used here that if a straight line is drawn through one of the nodes of a trinodal quartic and a break made at one of the other intersections, a unicursal quintic curve with a triple point results. This gives the two curves which are not directly obtainable from the conic and cubic.

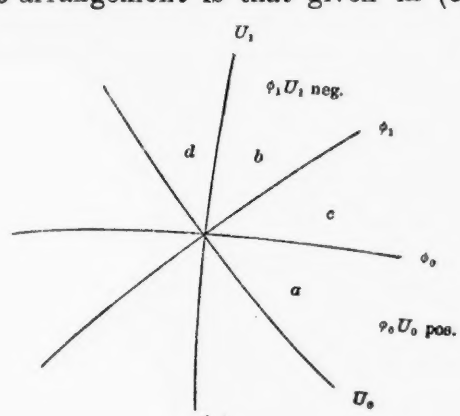
If $\phi_0 U_0$ and $\phi_1 U_1$ are divided at a given double point, as in figure (d_1), the



(d_1)

given point will always be a crunode no matter what value is given to λ; it may be positive or negative, large or small.

Suppose next the arrangement is that given in (e) and the products $\phi_0 U_0$

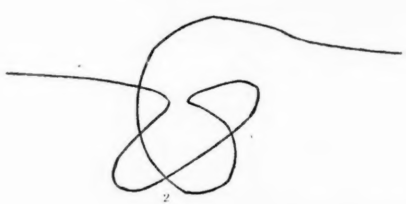


(e)

and $\phi_1 U_1$ are positive and negative in the compartments indicated. Then in the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$, if λ is negative, one branch of the curve is in (c) and the other in (d), while if λ is small and positive, the branches of the curve are both in (a), if large and positive, the branches are both in (b). In order that the curve shall change from one compartment to the other, the curve first has a cusp in compartment (a) (λ is now supposed a small positive quantity which is gradually growing larger), then there is a cusp in (b) and finally a crunode with the branches of the curve in compartment (b).

It might at first thought appear that it was possible for the curve to degenerate and have a tacnode instead of a cusp at the given point. Since the double points are fixed, there are only four possible ways of obtaining such a degenerate curve. To be definite, suppose the point considered is at 3, then the degenerate curve must be composed of a conic through the point 3, and any four of the remaining double points, together with a cubic which touches the conic at 3, has a node at the double point which does not lie on the conic, and passes through the four remaining double points. U_0 and U_1 each contain an arbitrary constant; clearly the quintic would not degenerate into the same conic and cubic independent of how these constants are chosen.

In case the part of the curve is like that given in figure (f), changing λ

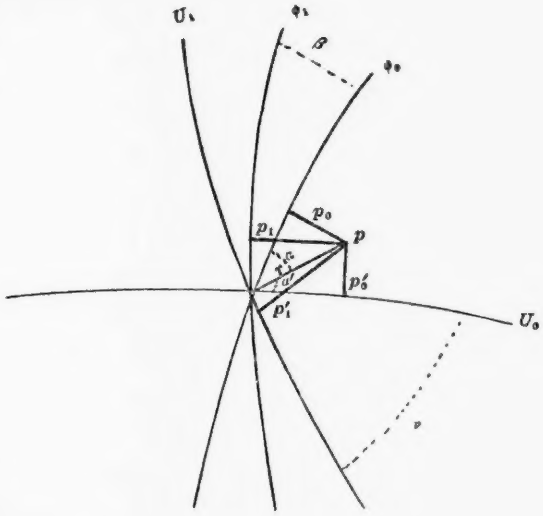


(f)

so as to have a cusp at 4 implies that 1, 2 and 3 have been successively crunodes, cusps, acnodes and possibly the curve has come back in the other compartment, so that one or more of them have been changed back first to cusps and then to crunodes.

It is now proposed to find an expression for the limiting value of λ. For a very small positive value of λ the two branches of the curve will be close to $\phi_0 U_0$, but as λ increases, the two branches approach each other until the crunode

is replaced by a cusp. Let p be such a point on the curve very close to the node, and let its coordinates be $x+\delta x,\ y+\delta y$.



Then $\phi_0 U_0 = \lambda \phi_1 U_1$,

$$\left(\delta x \frac{\partial \phi_0}{\partial x} + \delta y \frac{\partial \phi_0}{\partial y}\right)\left(\delta x \frac{\partial U_0}{\partial x} + \delta y \frac{\partial U_0}{\partial y}\right)$$

$$= \lambda \left(\delta x \frac{\partial \phi_1}{\partial x} + \delta y \frac{\partial \phi_1}{\partial y}\right)\left(\delta x \frac{\partial U_1}{\partial x} + \delta y \frac{\partial U_1}{\partial y}\right).$$

$$\lambda = \frac{\left(\delta x \frac{\partial \phi_0}{\partial x} + \delta y \frac{\partial \phi_0}{\partial y}\right)\left(\delta x \frac{\partial U_0}{\partial x} + \delta y \frac{\partial U_0}{\partial y}\right)}{\left(\delta x \frac{\partial \phi_1}{\partial x} + \delta y \frac{\partial \phi_1}{\partial y}\right)\left(\delta x \frac{\partial U_1}{\partial x} + \delta y \frac{\partial U_1}{\partial y}\right)}.$$

But

$$\delta x \frac{\partial \phi_0}{\partial x} + \delta y \frac{\partial \phi_0}{\partial y} = p_0 \sqrt{\left(\frac{\partial \phi_0}{\partial x}\right)^2 + \left(\frac{\partial \phi_0}{\partial y}\right)^2}.$$

$$\delta x \frac{\partial U_0}{\partial x} + \delta y \frac{\partial U_0}{\partial y} = p'_0 \sqrt{\left(\frac{\partial U_0}{\partial x}\right)^2 + \left(\frac{\partial U_0}{\partial y}\right)^2},$$

$$\delta x \frac{\partial \phi_1}{\partial y} + \delta y \frac{\partial \phi_1}{\partial y} = p_1 \sqrt{\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2},$$

$$\delta x \frac{\partial U_1}{\partial x} + \delta y \frac{\partial U_1}{\partial y} = p'_1 \sqrt{\left(\frac{\partial U_1}{\partial x}\right)^2 + \left(\frac{\partial U_1}{\partial y}\right)^2};$$

$$p_0 = r \sin \alpha, \qquad p'_0 = r \sin \alpha',$$
$$p_1 = r \sin(\alpha + \beta), \quad p'_1 = r \sin(\alpha' + \gamma).$$

The required value is

$$\lambda = \frac{\sin\alpha\,\sin\alpha'}{\sin(\alpha+\beta)\,\sin(\alpha'+\gamma)}\,\frac{\left[\left(\frac{\partial\phi_0}{\partial x}\right)^2+\left(\frac{\partial\phi_0}{\partial y}\right)^2\right]^{\frac{1}{2}}\cdot\left[\left(\frac{\partial U_0}{\partial x}\right)^2+\left(\frac{\partial U_0}{\partial y}\right)^2\right]^{\frac{1}{2}}}{\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial y}\right)^2\right]^{\frac{1}{2}}\cdot\left[\left(\frac{\partial U_1}{\partial x}\right)^2+\left(\frac{\partial U_1}{\partial y}\right)^2\right]^{\frac{1}{2}}},$$

where $\dfrac{\sin\alpha\,\sin\alpha'}{\sin(\alpha+\beta)\,\sin(\alpha'+\gamma)}$ must be taken so as to be a maximum.

To obtain the form of the curves having two or more consecutive nodes it is only necessary to take the conic and cubic with the desired number of consecutive intersections.

Let the origin be taken at a double point and let $y=0$ be a common tangent to the two branches. The curve will then have at least two consecutive double points at the origin. The following table gives the expansion of the branches and the corresponding singularity. These expansions are given in Salmon's Higher Plane Curves, p. 216, including the oscnode and tacnode cusp.

Nature of the singularity.	Expansion.
$\delta=4$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4, \\ y_2 = a_0x^2 + a_2x^3 + a_4'x^4, \end{cases}$
$\delta=3,\ \varkappa=1$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_5x^{\frac{9}{2}}, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_5'x^{\frac{9}{2}}, \end{cases}$
$\delta=5$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6'x^5, \end{cases}$
$\delta=4,\ \varkappa=1$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_7x^{\frac{11}{2}}, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_7'x^{\frac{11}{2}}, \end{cases}$
$\delta=6$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8x^6, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8'x^6, \end{cases}$
$\delta=5,\ \varkappa=1$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8x^6 + a_9x^{\frac{13}{2}}, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8x^6 + a_9'x^{\frac{13}{2}}. \end{cases}$

The curves having two or four imaginary nodes are obtained in exactly the same way as those having only real nodes, the only difference being that the conic is now taken so that in the first case two and in the second case four of its intersections with the cubic are imaginary.

As far as the sequence of double points is concerned, there is only one curve having six imaginary nodes. If in the equation

$$\alpha_1(\phi_1^2+\psi_1^2)+\varkappa_2\alpha_2(\phi_2^2+\psi_2^2)+\varkappa_3\alpha_3(\phi_3^2+\psi_3^2)=0,$$

$\varkappa_2$ and $\varkappa_3$ be taken very small, the form of the corresponding curve is very nearly the same as that of the straight line α_1.

2.—*Curves with a triple point.*

It will first be shown that the equation of any quintic curve with a triple point may be obtained from the special pencils

$$\alpha t^2 - \lambda\alpha_1 = 0, \quad \phi_1 t + \phi = 0, \tag{1}$$

or from

$$\alpha t^2 + \lambda\alpha_1 t - \alpha_1 = 0, \quad t\phi_1 + \phi = 0, \tag{2}$$

where t is a variable parameter, λ a constant, α and α_1 lines, ϕ and ϕ_1 conics.

On eliminating t from the first pair equations

$$\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0. \tag{3}$$

Similarly the equations (2) give

$$\alpha(\phi^2 - \phi_1^2) - \lambda\alpha_1\phi\phi_1 = 0. \tag{4}$$

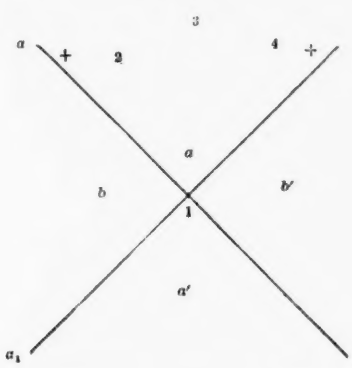
If in equations (3) α, $\lambda\alpha_1$, ϕ, ϕ_1, be replaced by $\alpha + i\lambda\alpha_1$, $\alpha - i\lambda\alpha_1$, $\phi + i\phi_1$, $\phi - i\phi_1$, respectively, the resulting equation reduces to

$$\alpha(\phi^2 - \phi_1^2) - 4\lambda\alpha_1\phi\phi_1 = 0, \tag{5}$$

which shows that (4) may be deduced from (3) or (3) from (4) by making the proper substitution.

It will now be shown that the equation $\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$ is the most general equation of a unicursal quintic curve having a triple point.

Let ϕ and ϕ_1 be two conics through the points 1, 2, 3, 4 and α, α_1 lines



through the point 1, λ a constant. Then the curve, whose equation is

$\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$, has a triple point at the point 1 and double points at 2, 3, 4. Moreover, each of the functions α, α_1, ϕ, ϕ_1 contains an undetermined constant which, together with λ, makes the number in the above equation 5, exactly that contained in the general equation of a unicursal quintic having a triple point.

Suppose in the above figure that there is an ordinary point of the curve in compartment a. Then the whole curve must lie in a (and a'), with the exception of possible acnodes. For, in order that a point shall be on the curve, $\alpha\phi^2$ must be equal to $\lambda\alpha_1\phi_1^2$ at the given point, but in crossing α or α_1 one of these terms changes sign and not the other, hence the only point at which it is possible for the curve to cross the lines α, α_1 is at their intersection. For the sake of definiteness, let λ be positive and let the positive sides of α and α_1 be taken as in the figure. Then if 2, 3 and 4 be taken in compartment a (or a'), they will be crunodes on the given quintic. By taking any or all of them in the compartment b (or b) or by simply changing the sign of λ, they become acnodes.

Take α through the point 2. Then, from the equation

$$\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0,$$

it is clear that since α and ϕ are both zero at 2, while α_1 is finite, points on the curve in the neighborhood of 2 must lie very close to ϕ_1. Moreover, since the curve has a double point at 2 and does not cross α, there must be a cusp at the given point with the tangent to ϕ_1 as the cuspidal tangent. By taking α_1 through one of the two remaining double points, it is possible to obtain another cusp.

If the conics intersect in only two real points, then the case arises in which the curve has two imaginary double points. These double points may become imaginary cusps, which would be the case in equation (4) if the conics ϕ and ϕ_1 are circles.

By taking conics having two consecutive intersections, a tacnode results, and if α is taken through the given point, the cusp is of the second kind. The conics might also be taken, having three or even all four of their intersections consecutive.

The curves represented by the equation $\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$ were drawn by fixing:

1st. The four points of intersection of ϕ and ϕ_1.

2d. The lines α and α_1.

3d. The three tangents at the triple point.

This makes a sufficient number of conditions to completely define the curve.

In drawing the curves, nothing more than the relative positions of these elements was considered, so that in a few cases the same initial conditions give rise to more than one curve. In such cases there would be some point at which the curve could move in different ways without violating the initial conditions.

The three double points were generally taken on the same side of the triple point. Of course, it is always possible to project the given curve into a curve in which this is the case, but it may not be possible to do so without altering the number of infinite branches. For instance, figs. 85, 112, 115, have the same relative positions of nodes, tangents at the triple point and tangents to the curve from the triple point. But if 85 is projected into a curve where the double points are situated as in 112, it will have at least three infinite branches: 115 may have its double points either way and still have only one infinite branch.

In some cases as for instance in figure 84 the curve may be tangent to α on either side of the triple point. This simply amounts to pushing the inflexion along the curve through the triple point, and although such curves are projectively distinct there is not enough difference to require two figures.

In drawing the curves represented by equation (4), the lines α and α_1 were not drawn but the tangents at the triple point were fixed as before and in addition the part of the curve going to infinity was taken in the various possible ways (joining two nodes, node to triple point, triple point to triple point). Use was also made of the fact that these curves can have no real tangents from the triple point, except those at the triple point.

The figures are only intended to give the general form of the curves. In some cases they do not show the total number of inflexions. It is then supposed that the curve has isolated double tangents.

The existence of these various types is also made plausible from Art. 1 by allowing three distinct double points to approach coincidence in various ways. The equations can also be determined exactly as in the preceding case, but it was frequently easier to follow the curve by present method.

3.—*Curves with a fourfold point.*

The case where the double points are all coincident would give but a single type according to Meyer's classification, but for these curves a different plan will be adopted, namely, the configuration of the fourfold point and the compartments

formed by loops or the infinite branches will form the basis of classification. Here the reality of the inflexions will be considered.

The curve may be considered as generated by the intersection of corresponding rays of the pencils

$$a_0t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0$$

and

$$b_0t + b_1 = 0,$$

in which $a_i = 0$ and $b_i = 0$ are the equations of straight lines and t is a parameter. The vertex of the linear pencil is a fourfold point on the quintic: it will be taken as the origin so that the equation of the curve may be written $u_5 + u_4 = 0$. The factors of u_5 determine the directions of the infinite branches and the factors of u_4 determine the tangents at the origin. These tangents divide the plane into compartments such that the curve in passing from one compartment to another must either pass through the origin or through infinity. The nature of the fourfold point will depend on the form of u_4.

The following table gives Plücker's numbers and also $I + 2T$ for the various forms of the fourfold point.

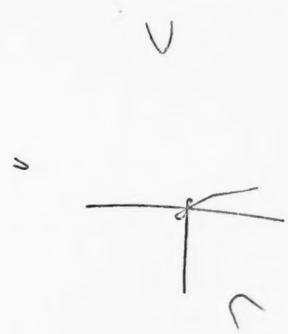
Factors of u_4	n	δ	$\varkappa$	τ	i	$I+2T$	Figures.
$m_1 m_2 m_3 m_4$	8	6	0	12	9	3	134, 135, 136, 137,
$m_1^2 m_2 m_3$	7	5	1	8	7	3	138, 139, 140, 141, 142,
$m_1^2 m_2^2$	6	4	2	5	5	3	143, 144, 145,
$m_1^3 m_2$	6	4	2	5	5	3	146, 147, 148,
m_1^4	5	3	3	3	3	3	149,
$m_1 m_2 (m_3^2 + m_4^2)$	8	6	0	12	9	5	150, 151, 152, 153, 154, 155,
$m_1^2 (m_2^2 + m_3^2)$	7	5	1	8	7	5	156, 157, 158,
$(m_1^2 + m_2^2)(m_3^2 + m_4^2)$	8	6	0	12	9	5	159, 160,
$(m_1^2 + m_2^2)^2$	6	4	2	5	5	5	161, 162.

In case the factors of U_4 are real and distinct, there are the four types of curves represented in the first four figures. That the three inflexions may all be in one compartment formed by two tangents at the multiple point may be

seen by considering a curve having five infinite branches in the given compartment as in the accompanying figure.

When the asymptotes are drawn, it is clear that there must be an inflexion somewhere along the curve either at a or a', one at b or b', and one at c or c'.

By taking a vanishing line close to the common tangent of E and F, this figure projects into figure 135. By taking four infinite branches in a given com-



partment, it is seen that the resulting curve has a loop which cuts the vanishing

line in four points. The curve, therefore, has a loop with a bay. Figure 136 differs from figure (g) only in the position of the vanishing line.

The equations have not been written out for each case as they follow directly; for instance, the equation of figure 137 is

$$abc(d^2+f^2)-\lambda\alpha\beta\gamma\delta=0,$$

d and f are lines and λ is a parameter; a, b, c are lines through the origin parallel to the asymptotes, α, β, γ, δ are the four tangents to the curve at the fourfold point. The equation corresponding to a curve like figure 136 would not follow quite so directly, as it would be necessary to take the equation corresponding to figure (g), and perform on it the transformation corresponding to the projection of figure (g) into figure 136. No figures are drawn where the curve has five asymptotes, as such a curve can always be projected into one of the given figures.

In figures 143 to 148 the fourfold point counts as two cusps and four crunodes. In the first three figures the form of the curve indicates this, while in the last three the origin has the appearance of an ordinary double point. The presence of evanescent loops being shown by deformation.

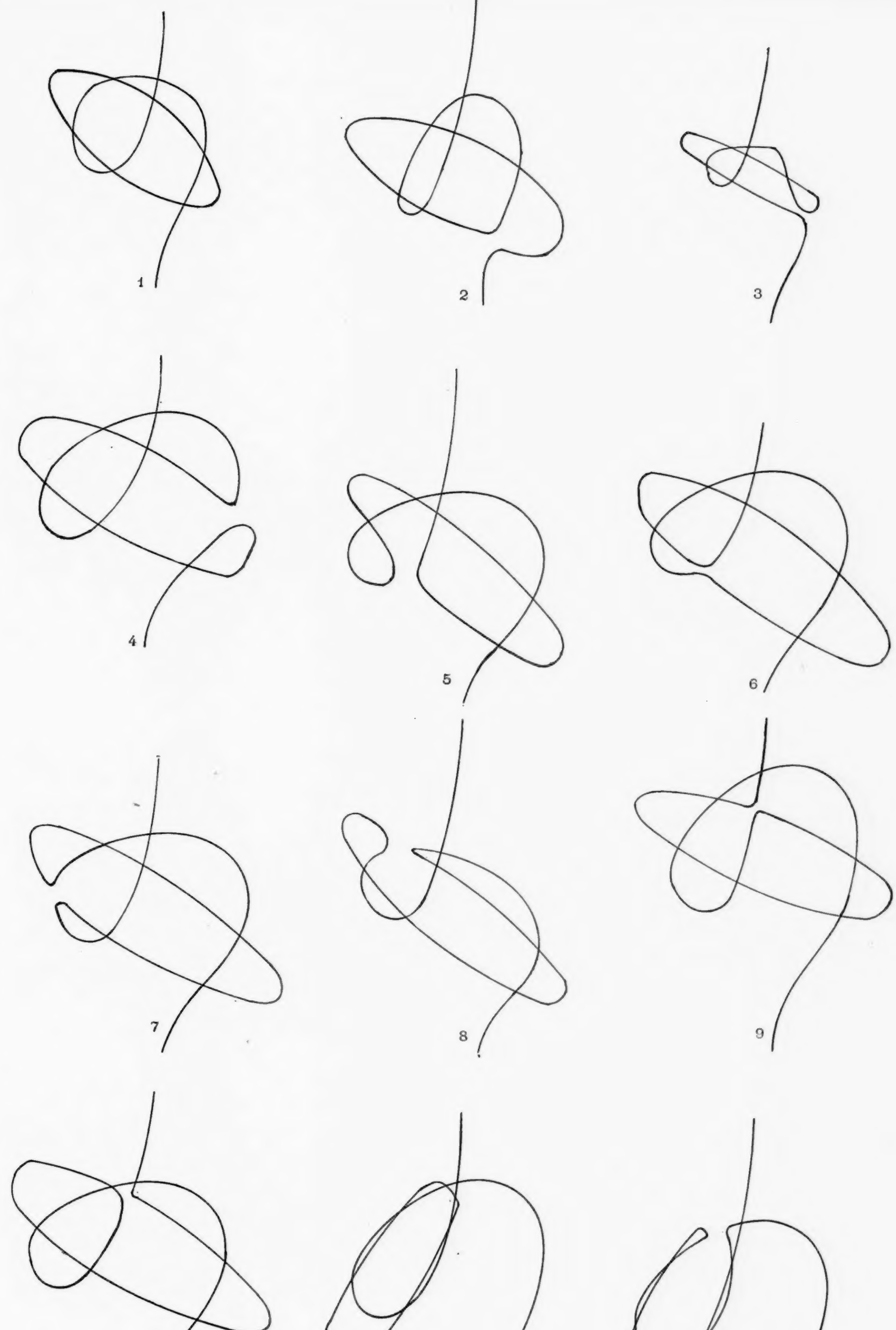
Figure 155 differs from the others in that it contains a double bay. The possibility of this is readily seen from 141 (provided one of the loops in 141 had a bay) which was obtained from figure 136 by making two of the factors of u_4 equal. If now u_4 be changed still further so that these two factors become conjugate imaginaries, the cusp is replaced by an acnode and the curve takes the form of figure 155.*

When the curve has an isolated fourfold point, it is not possible to have fewer than three inflexions, as was shown by Möbius: "No odd branch containing no real point singularities can have fewer than three real inflexions."

It may be mentioned that there exists for every order n a curve corresponding to figure 137, i. e., having an $n-1$ fold point and $n-2$ infinite branches. For, take a curve having an $n-1$ fold point and whose equation can therefore be written $u_n+u_{n-1}=0$. Take the factors of u_{n-1} real and distinct. The lines will divide the plane in $n-1$ compartments. If the lines represented by u_n are

* There should be given two figures whose form is the same as 141 excepting as regards inflexions, the one having a bay on one loop and the other having 3 real inflexions but none on the loop.

PLATE I.



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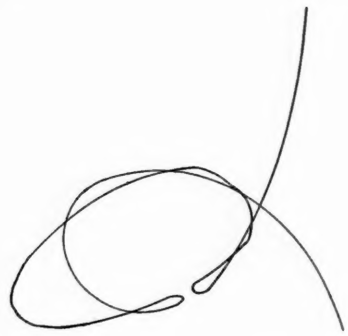

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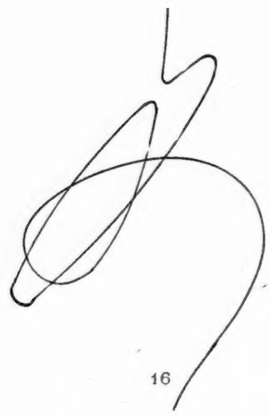

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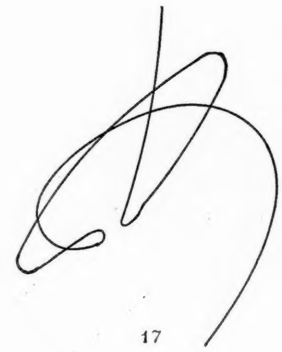

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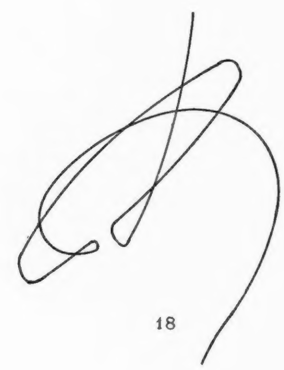

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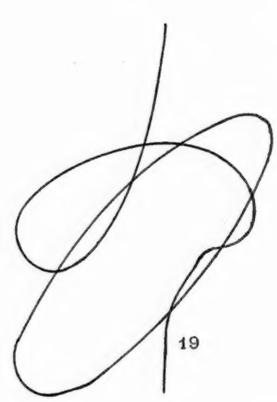

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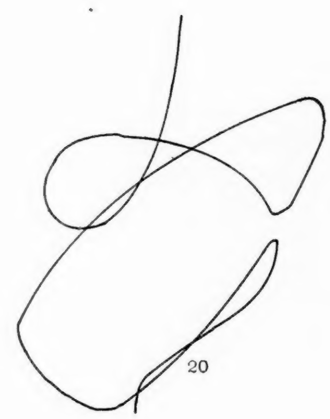

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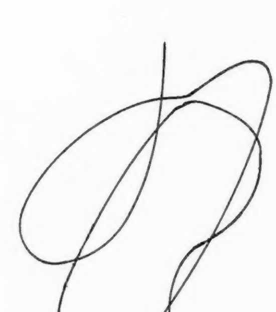

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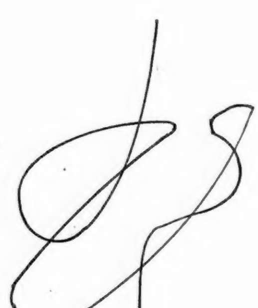

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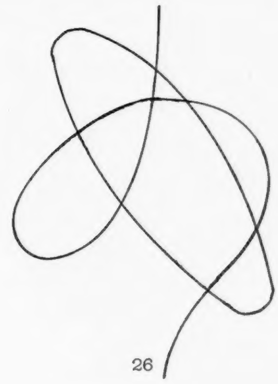
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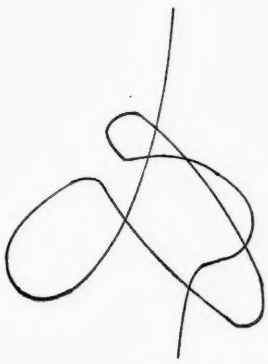
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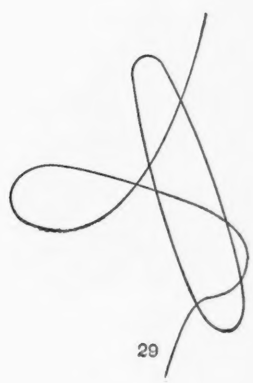
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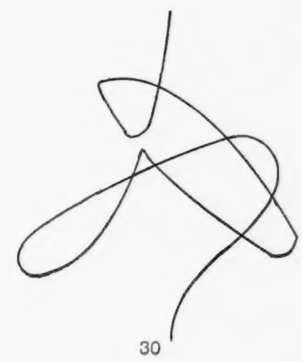
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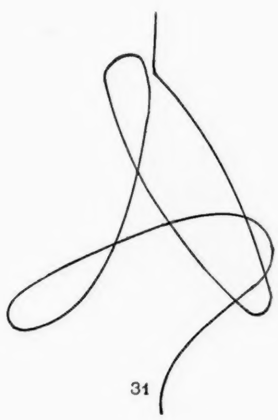
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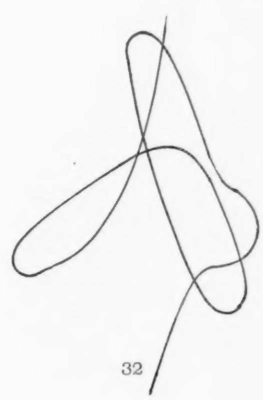
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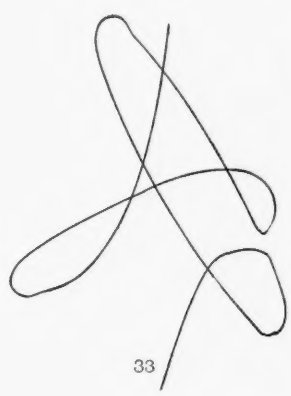
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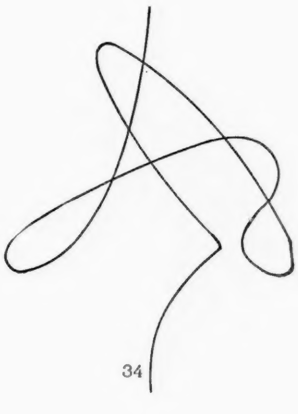
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PLATE IV.

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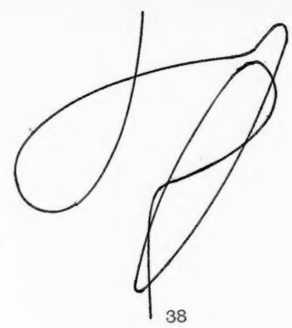

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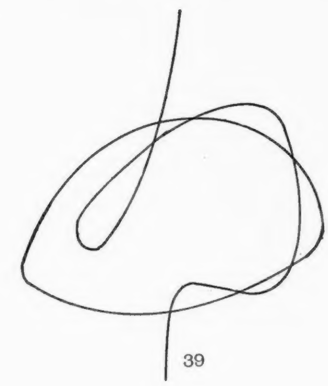

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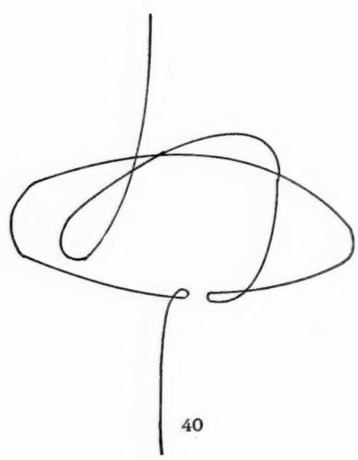

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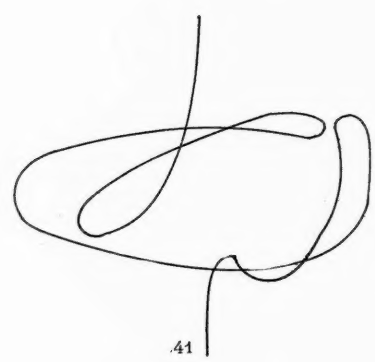

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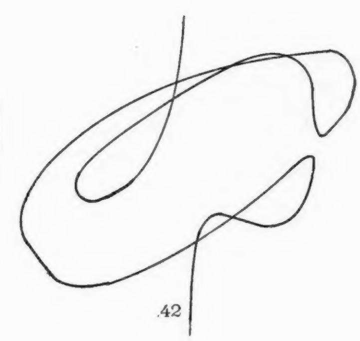

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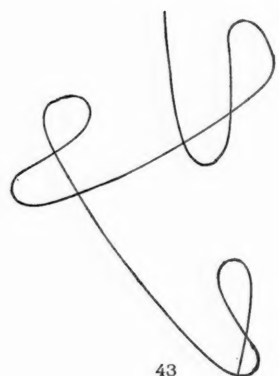

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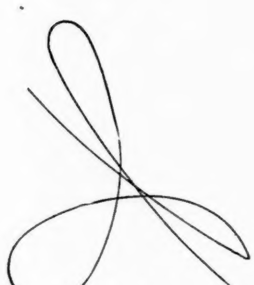

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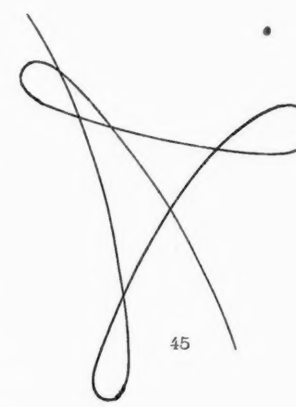

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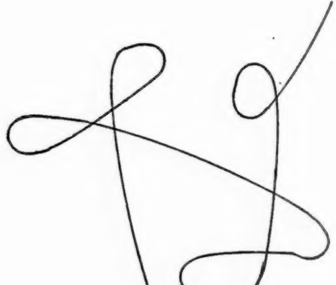

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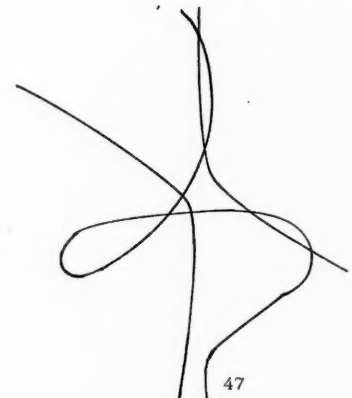

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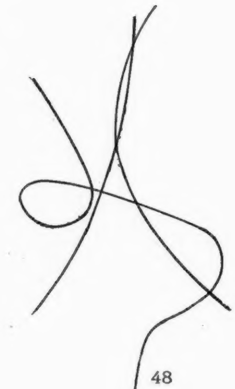

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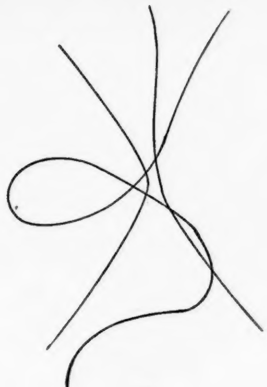

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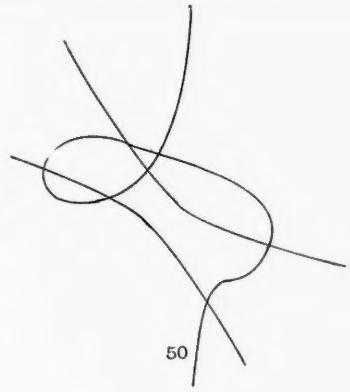

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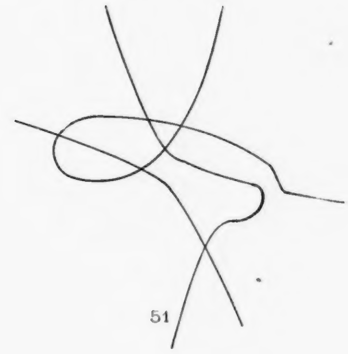

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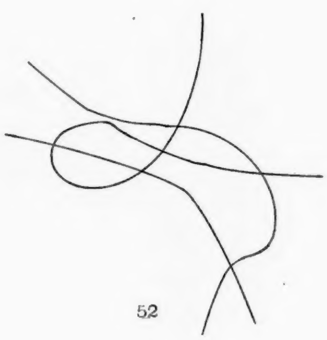

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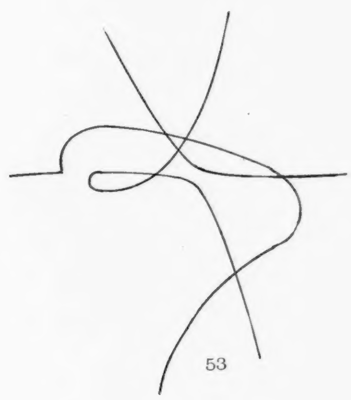

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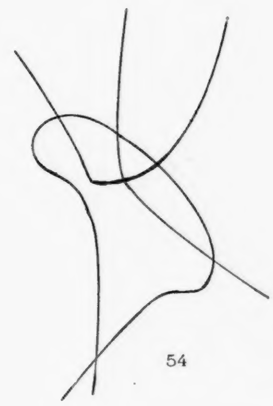

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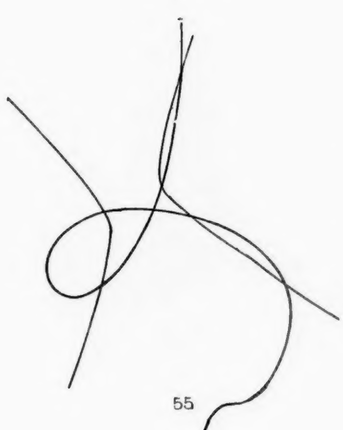

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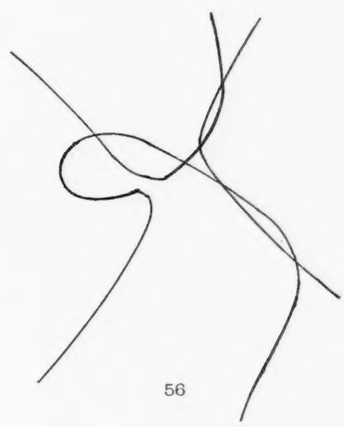

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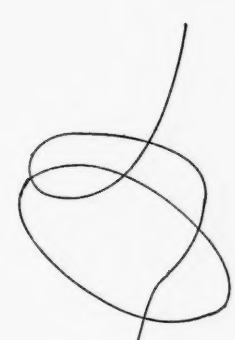

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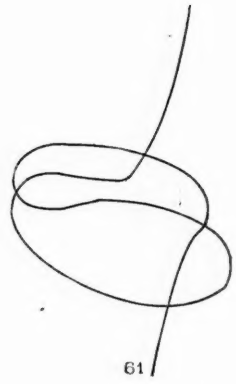
PLATE VI.



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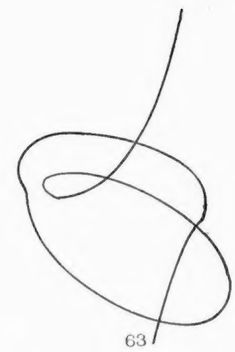

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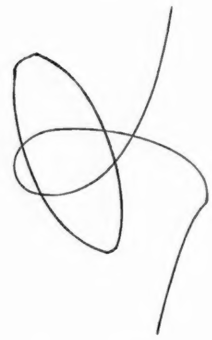

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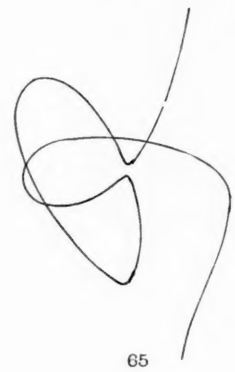

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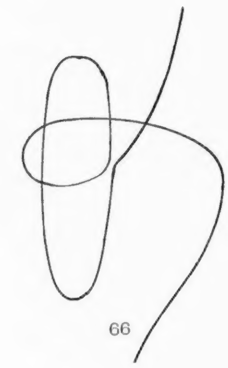

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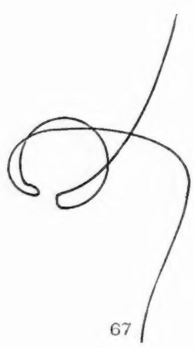

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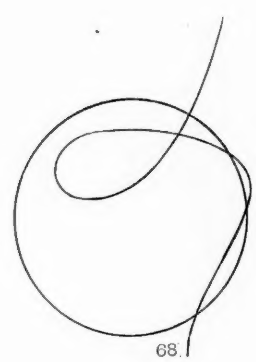

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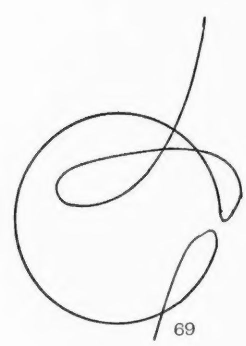

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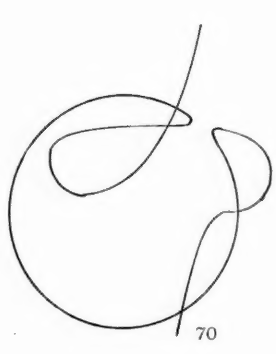

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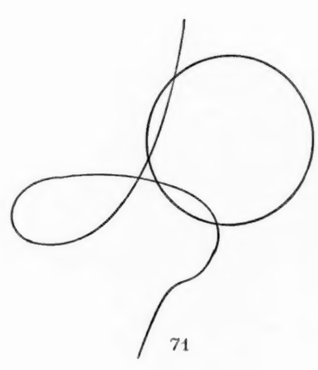

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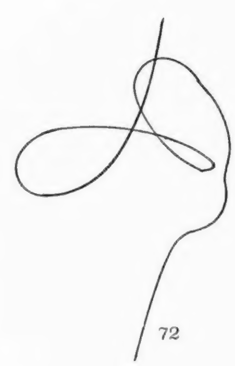

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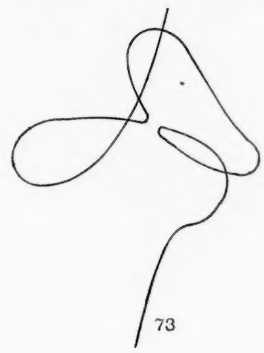

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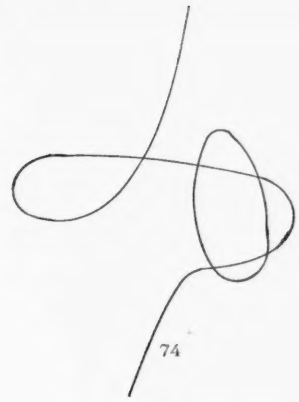

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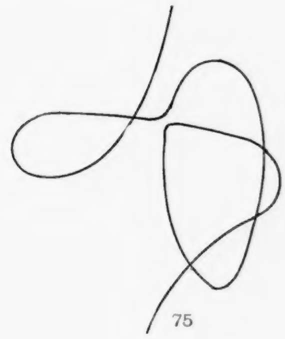

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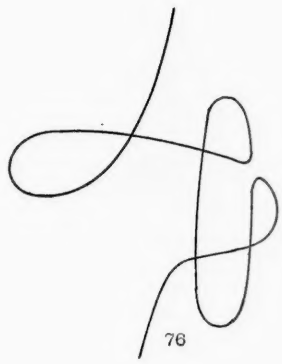

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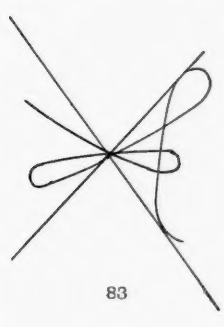

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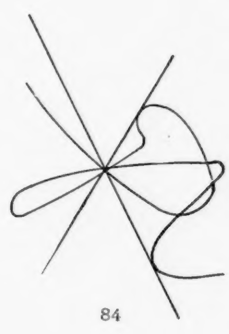

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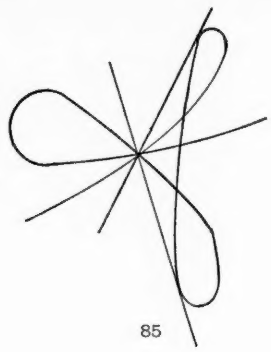

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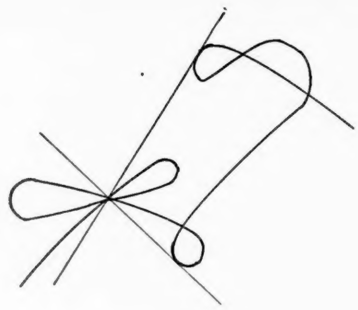

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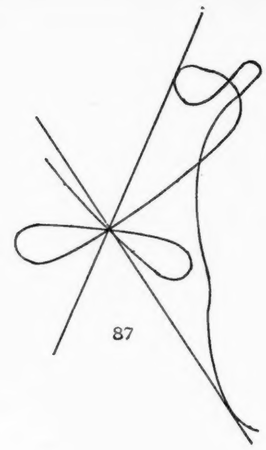

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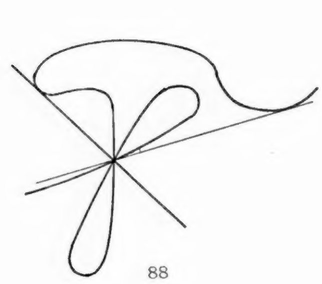

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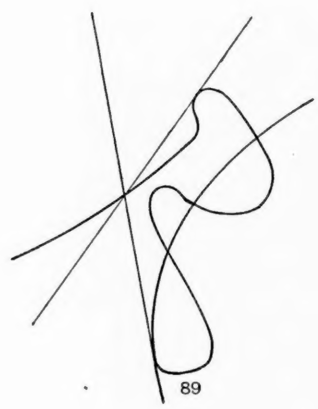

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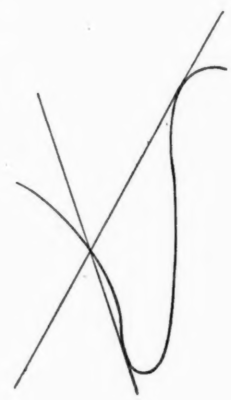

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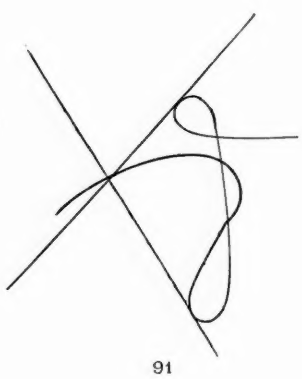

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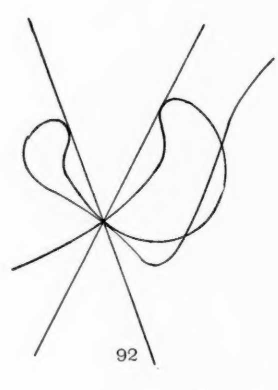

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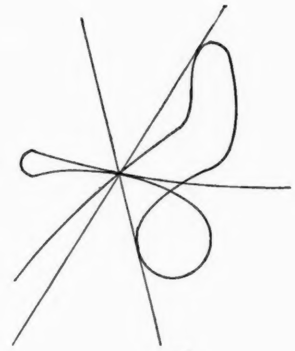

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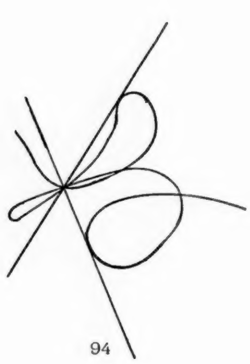

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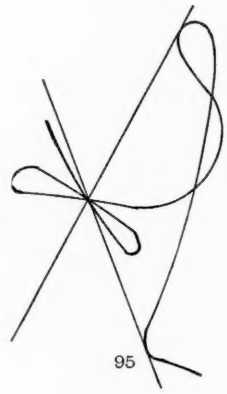

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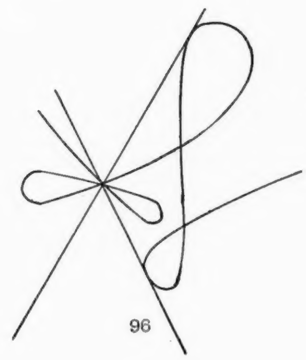

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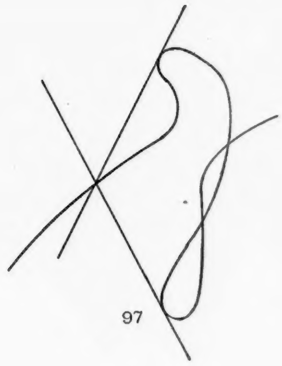
PLATE IX.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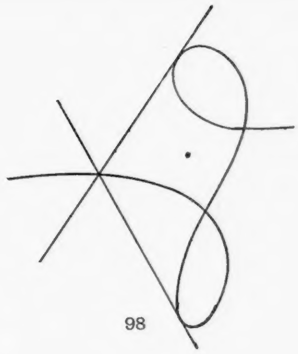
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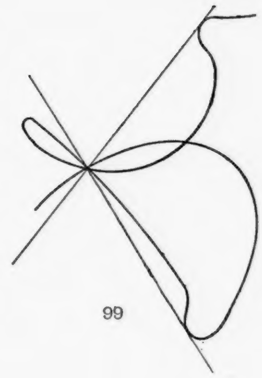
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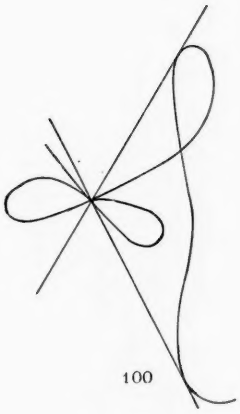
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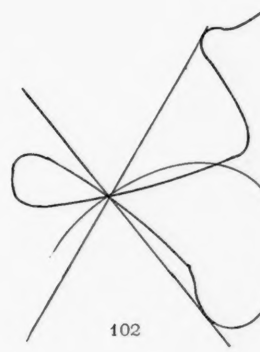
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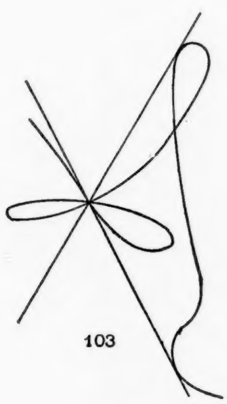
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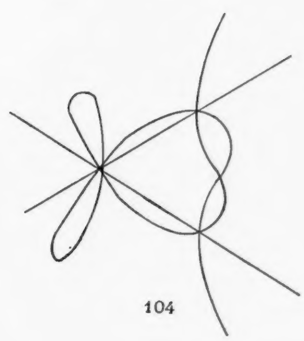
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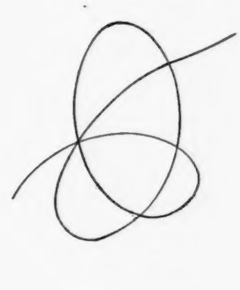
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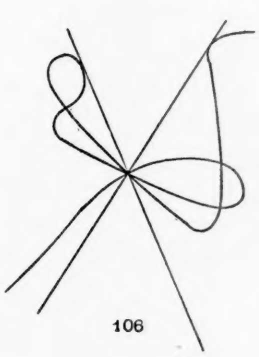
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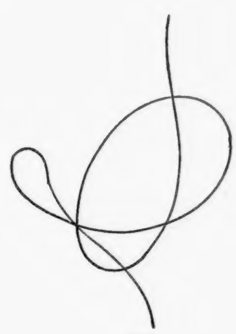
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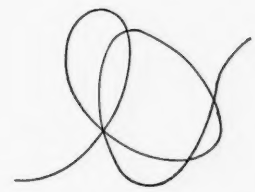
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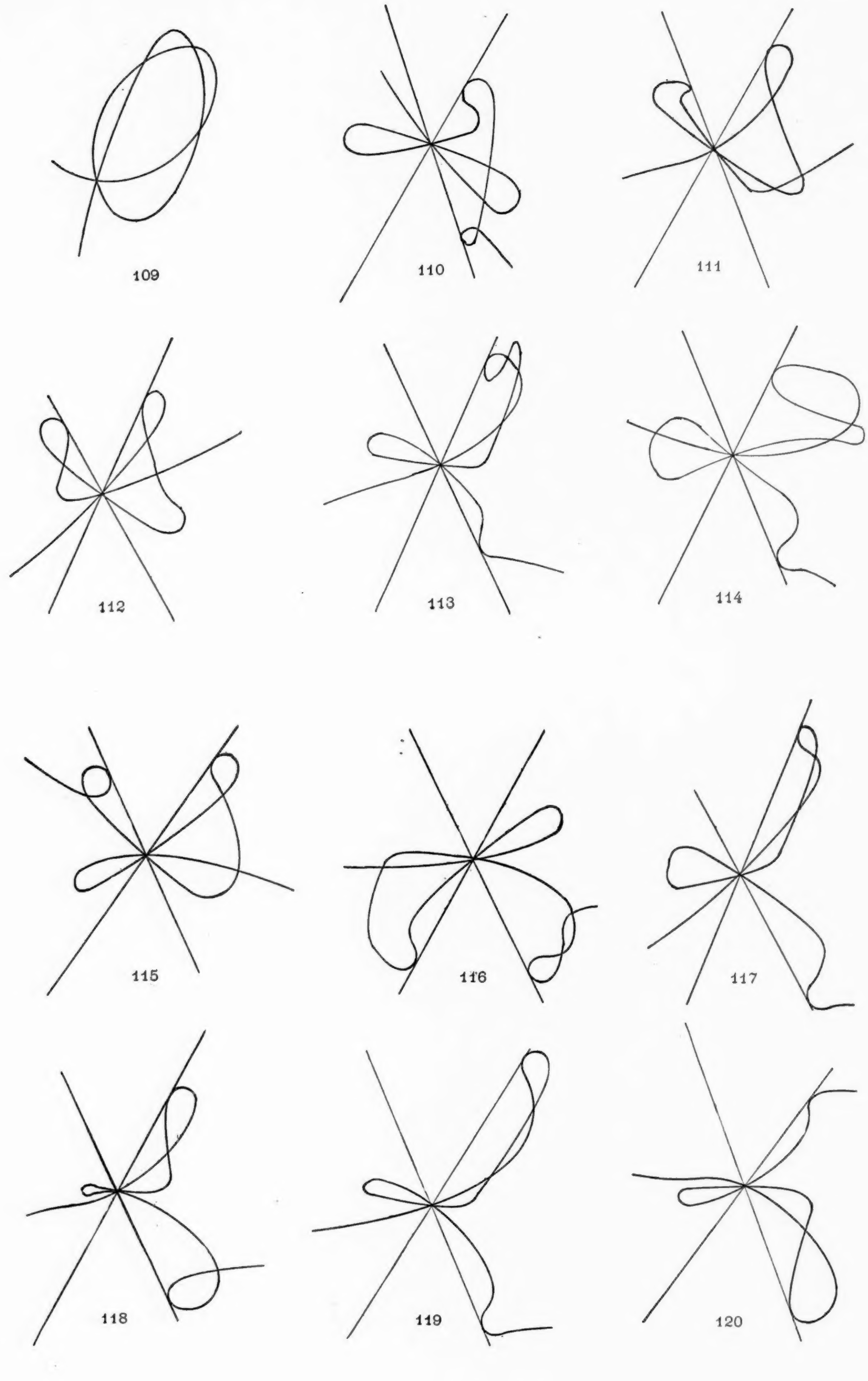
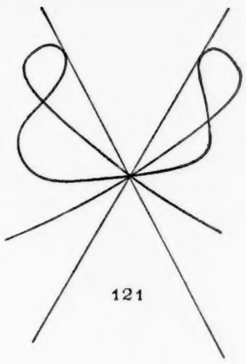


PLATE XI.



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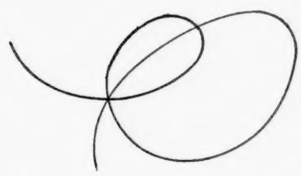

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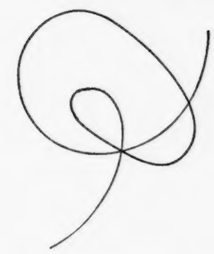

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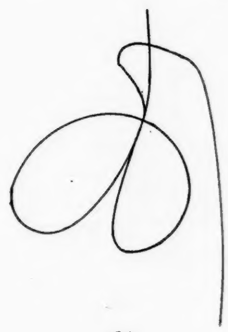

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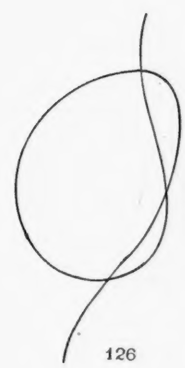

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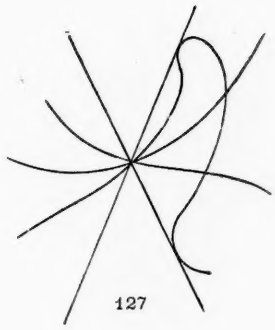

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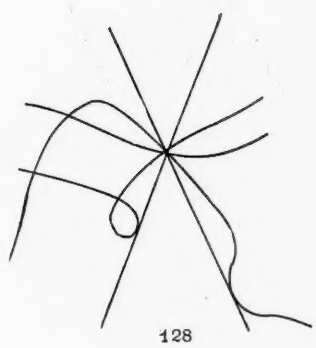

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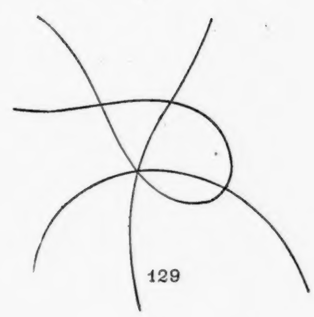

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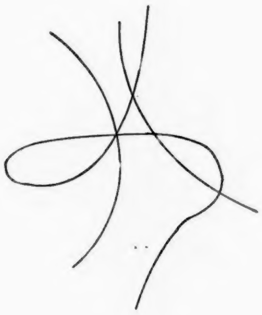

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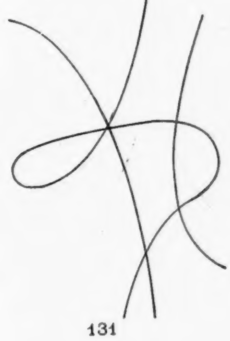

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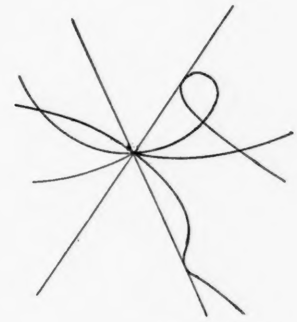

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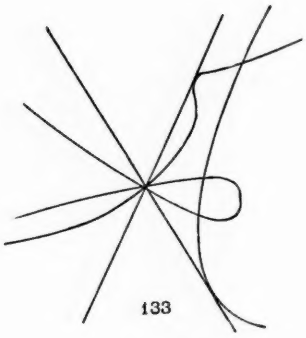
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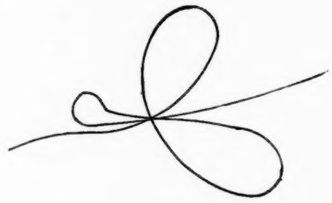
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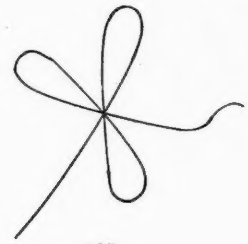
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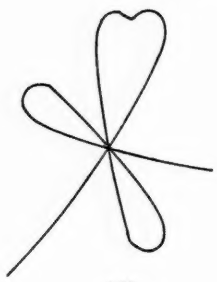
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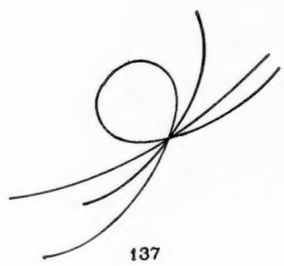
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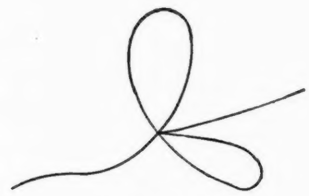
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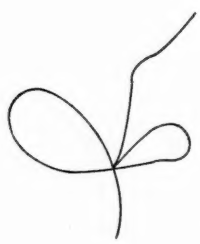
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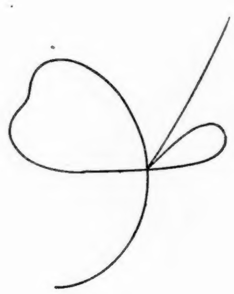
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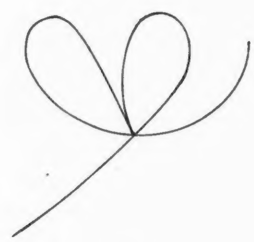
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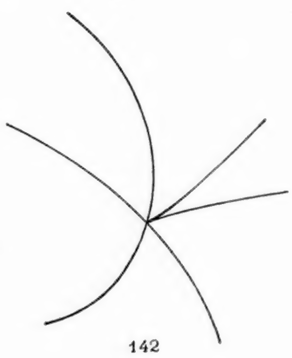
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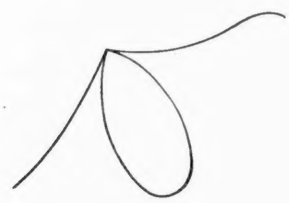
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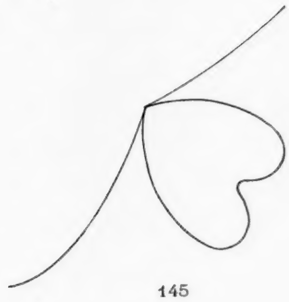

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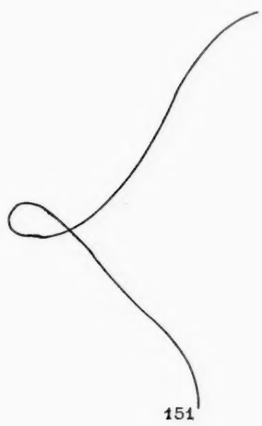

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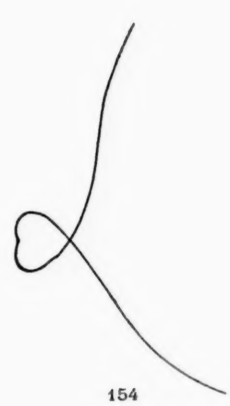

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PLATE XIV.


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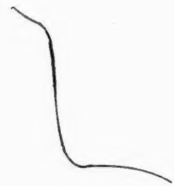
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taken so that one lies in each but one of the compartments, the resulting curve cannot be projected into one with fewer than $n-2$ infinite branches, as the given curve has $n-2$ branches that extend across the plane. The two remaining factors of u_n may be taken conjugate imaginary.

CORNELL UNIVERSITY, *June*, 1902.

*Determination of the Algebraic Curves whose Polar Conics are Parabolas.**

By Edward Kasner.

Among the curves of n^{th} order, one of the simplest classes, usually considered in connection with the theory of equations, is composed of those curves whose cartesian equation may be put into the form

$$y = a_0 x^n + a_1 x^{n-1} + \dots + a_n.$$

It is easily shown that the polar conic of any point in the plane with respect to such a curve is a parabola. The object of this note is to prove that this property is characteristic, i. e., that if any (non-decomposable) curve has the property stated, it may be reduced to the above form. The proof hinges upon the discussion of a partial differential equation, which is of interest also in connection with a class of developable surfaces.

Since the question considered belongs to metric geometry, it will be convenient to first derive the equations for polar curves in rectangular, instead of the usual homogeneous coordinates.

The first polar of the point x', y' with respect to the curve of n^{th} order $\phi(x, y) = 0$ is easily shown to be

$$x'\phi_x + y'\phi_y + n\phi - x\phi_x - y\phi_y = 0; \tag{1}$$

and the second polar is

$$\left.\begin{aligned} &x'^2\phi_{xx} + 2x'y'\phi_{xy} + y'^2\phi_{yy} \\ &+ 2x'\{(n-1)\phi_x - (x\phi_{xx} + y\phi_{xy})\} + 2y'\{(n-1)\phi_y - (x\phi_{xy} + y\phi_{yy})\} \\ &+ \{n(n-1)\phi - 2(n-1)(x\phi_x + y\phi_y) + x^2\phi_{xx} + 2xy\phi_{xy} + y^2\phi_{yy}\} = 0.\end{aligned}\right\} \tag{2}$$

---

* Read before the American Mathematical Society, Dec. 28, 1901.

If x', y' are regarded as current coordinates, (6) represents the polar line of the point x, y and (7) represents the polar conic.

The polar conic will be a parabola provided

$$\Delta \equiv \phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0. \tag{3}$$

Hence in connection with the general curve of n^{th} order we have a (metrically) related curve* $\Delta = 0$, of order $2(n-1)$, the locus of points whose polar conics are parabolas.

The problem before us may now be stated: *find the curves* $\phi = 0$ *for which* Δ *vanishes identically*, i. e., determine the rational integral solutions, of the n^{th} degree, of the partial differential equation *

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0. \tag{4}$$

If ϕ is written in the form

$$\phi = u + R,$$

where u includes all the terms of n^{th} order and R the terms of lower order, the substitution in (4) gives as a necessary condition

$$u_{xx}u_{yy} - u_{xy}^2 = 0.$$

Since u is a binary form this indicates that it must be the n^{th} power of a linear form. Without loss of generality, we may write

$$\phi = x^n + R, \tag{5}$$

where R includes only terms whose degree is less than n.

The next step in the discussion is to show that if a function of form (5) satisfies (4), then R must be of the form

$$R = f(x) + cy, \tag{6}$$

where f is a polynomial in x of degree $n-1$ and c is a constant.

---

* From the form of its equation, this might be termed the metric or cartesian Hessian of the original curve. Cf. C. A. Scott, "Note on the Real Inflexions of Plane Curves" (Transactions of the American Mathematical Society, Vol. 3, 1902, p. 398), where, more generally, the locus of points whose polar conics touch a given line is considered, and termed the *diacritic* of the line with respect to the curve.

† This is a Monge equation whose general solution may be obtained without difficulty; but this does not help in finding the rational integral solutions here required.

For the proof we employ the method of mathematical induction. We assume then that the result is correct when the order is $n-1$, and proceed to prove it for the order n.

In the first place, if the curve $\phi = 0$ belongs to the class considered, that is, if all its polar conics are parabolas, the same is evidently true of the polar curves

$$\xi\phi_x + \eta\phi_y + n\phi - x\phi_x - y\phi_y = 0 .$$

These are curves of order $n-1$ to which the assumption may be applied. Take first the point $\xi = \infty$, $\eta = 0$, whose polar curve is

$$\phi_x \equiv nx^{n-1} + R_x = 0 .$$

By the assumption made, the last term is of the form

$$R_x = f_{n-2}(x) + ky .$$

The integration of this gives

$$R = f(x) + kxy + g(y), \tag{7}$$

where f and g are of degree $n-1$.

Introducing this value in (5), and substituting in (4), the result should be an identity. Equating, in particular, the coefficient of ξ and the absolute term to zero, we obtain

$$\{n(n-1)(n-2)x^{n-3} + f'''\}(ng - yg')'' = 0, \tag{8}$$

$$(nf - xf')''(ng - yg')'' = (n-2)^2 k^2, \tag{9}$$

where the accents indicate differentiation with respect to the variable involved.

Disregarding the cases $n = 1$ or 2, which may be considered by themselves without difficulty and taken as the starting point of the induction, it follows that the second factor of (8) must vanish. For since the degree of f is at most $n-1$, it is evident that the first factor cannot vanish. We have then

$$(ng - yg')'' = 0 ,$$

which, integrated twice, gives

$$ng - yg' = Ly + M .$$

We seek now for the integral of this equation whose degree does not exceed $n-1$. For this purpose substitute

$$g = b_0 y^{n-1} + b_1 y^{n-2} + \dots + b_{n-1},$$

obtaining the conditions

$$b_0 = b_1 = \dots = b_{n-3} = 0, \quad b_{n-2} = \frac{L}{n-1}, \quad b_{n-1} = \frac{M}{n-1}.$$

Hence g is of the form

$$g = ly + m.$$

Substituting this value of g in (9), it follows that $k = 0$. Hence (7) reduces to the form (6), and the latter is justified. This completes the induction.

From (5) and (6), the curve $\phi = 0$ may be written

$$x^n + f(x) + cy = 0. \tag{10}$$

If $c = 0$, this represents merely a set of parallel straight lines. Disregarding this trivial case, the equation may be written

$$y = F(x),$$

where F is a polynomial of n^{th} order.

If a curve of the n^{th} order has the property that all its polar conics are parabolas, then either it consists of n parallel straight lines, or it is of the form

$$y = a_0 x^n + a_1 x^{n-1} + \dots + a_n. \tag{11}$$

Conversely, both of these classes have the property in question, the polar conics in the first case being pairs of parallel lines, while in the second they are proper parabolas.

Incidentally, we have obtained, essentially, the rational integral solution of the differential equation (4). It is merely necessary to free the form (11) from the special choice of axes, introducing, for example, $ax + by$ and $by - ax$ instead of x and y respectively. *The general rational integral solution, of the n^{th} degree, of equation* (4) is

$$\phi = (hx + ky)^n + A_1 (hx + ky)^{n-1} + \dots + A_{n-2}(hx + ky)^2 + A_{n-1}x + A_n y + A_{n+1}, \tag{12}$$

thus involving $n + 3$ arbitrary constants.

If this is equated to zero, the result is the general curve having the property in question; it involves $n + 2$ parameters.*

* The degenerate case arises when the constants are connected by the relation $hA_n - kA_{n-1} = 0$.

For the proof we employ the method of mathematical induction. We assume then that the result is correct when the order is $n-1$, and proceed to prove it for the order n.

In the first place, if the curve $\phi = 0$ belongs to the class considered, that is, if all its polar conics are parabolas, the same is evidently true of the polar curves

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$$g = b_0 y^{n-1} + b_1 y^{n-2} + \dots + b_{n-1},$$

obtaining the conditions

$$b_0 = b_1 = \dots = b_{n-3} = 0, \quad b_{n-2} = \frac{L}{n-1}, \quad b_{n-1} = \frac{M}{n-1}.$$

Hence g is of the form

$$g = ly + m.$$

Substituting this value of g in (9), it follows that $k = 0$. Hence (7) reduces to the form (6), and the latter is justified. This completes the induction.

From (5) and (6), the curve $\phi = 0$ may be written

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If a curve of the n^{th} order has the property that all its polar conics are parabolas, then either it consists of n parallel straight lines, or it is of the form

$$y = a_0 x^n + a_1 x^{n-1} + \dots + a_n. \tag{11}$$

Conversely, both of these classes have the property in question, the polar conics in the first case being pairs of parallel lines, while in the second they are proper parabolas.

Incidentally, we have obtained, essentially, the rational integral solution of the differential equation (4). It is merely necessary to free the form (11) from the special choice of axes, introducing, for example, $ax + by$ and $by - ax$ instead of x and y respectively. *The general rational integral solution, of the n^{th} degree, of equation* (4) is

$$\phi = (hx + ky)^n + A_1 (hx + ky)^{n-1} + \dots + A_{n-2}(hx + ky)^2 + A_{n-1}x + A_n y + A_{n+1}, \tag{12}$$

thus involving $n + 3$ arbitrary constants.

If this is equated to zero, the result is the general curve having the property in question; it involves $n + 2$ parameters.*

* The degenerate case arises when the constants are connected by the relation $hA_n - kA_{n-1} = 0$.

The solution (12) may be applied to a question concerning developable surfaces. Consider the surfaces whose equation may be reduced to the form

$$z = F(x, y), \tag{13}$$

where F is a rational integral function of n^{th} order.

Such a surface will be developable when and only when, F is a solution of (9) *and hence of the form* (12). In this case the level curves cut out by the planes $z =$ constant will belong the class treated. Conversely if one (and hence all) of the level curves of the surface (13) belong to the class treated, then the surface is developable. The developable surfaces thus obtained are cylindrical.

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On Certain Conics Connected with Trinodal Quartics.

By A. B. Basset.

1. It is well known that three conics can be described which respectively pass, (i), through the six points in which the nodal tangents intersect the quartic, (ii), through the six points of contact of the tangents drawn from the nodes, (iii), through the six points of inflexion; also, that each of the three conics intersect the quartic at two points S, S', which will be called the S points. A somewhat lengthy proof of the first theorem is given in §194 of my book on "Cubic and Quartic Curves;" and concise proofs of the three theorems, together with the equations of the three conics, appear to be a *desideratum*.

There is also a fourth conic which passes through the six Q points of a trinodal quartic. The equation of this conic will be found, and it will be shown to pass through two points T, T', which will be called the T points, which are the two remaining points in which the line SS' cuts the quartic. The T points are also points of importance in the theory of trinodal quartics.

Let the equation of the quartic be

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 + 2\alpha\beta\gamma\,(l\alpha + m\beta + n\gamma) = 0, \tag{1}$$

to which form every trinodal quartic can be reduced by projection. Let

$$\sigma = l\beta\gamma + m\gamma\alpha + n\alpha\beta, \tag{2}$$

$$\tau = \beta\gamma/l + \gamma\alpha/m + \alpha\beta/n, \tag{3}$$

$$u = k_1\alpha + k_2\beta + k_3\gamma, \tag{4}$$

where

$$k_1 = m/n + n/m - 2l, \tag{5}$$

with similar expressions for k_2, k_3. Then (1) may be written in the form

$$\sigma\tau - \alpha\beta\gamma u = 0. \tag{6}$$

Equation (6) shows that the line $u = 0$ intersects the quartic in four points, two of which, S, S', lie in the conic $\sigma = 0$, whilst the remaining two, T, T', lie in the conic $\tau = 0$. And if §194 of my book, $2l$, $2m$, $2n$ be written for l, m, n,

it can easily be shown that the equation of the conic may be written in the form

$$\Delta\sigma + u(\alpha/l + \beta/m + \gamma/n) = 0, \tag{7}$$

where

$$\Delta = -4 - \frac{1}{l^2} - \frac{1}{m^2} - \frac{1}{n_2} + \frac{4(l^2 + m^2 + n^2) + 1}{2lmn},$$

which shows that the conic (7), which passes through the six points in which the nodal tangents intersect the quartic, also passes through the S points.

2. *To find the equation of the conic which passes through the six points of contact of the tangents drawn from the nodes.*

Equation (1) may be written in the form

$$\alpha^2\{\beta^2 + \gamma^2 + 2l\beta\gamma - (m\beta + n\gamma)^2\} + \{\alpha(m\beta + n\gamma) + \beta\gamma\}^2 = 0, \tag{8}$$

which shows that the equation of the tangents drawn from the node A is

$$\beta^2 + \gamma^2 + 2l\beta\gamma - (m\beta + n\gamma)^2 = 0, \tag{9}$$

and that the points of contact lie in the conic

$$\alpha(m\beta + n\gamma) + \beta\gamma = 0, \tag{10}$$

from which it is easily shown that the equation of the chord of contact of the tangents is

$$k_1\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma = 0, \tag{11}$$

The equation

$$\sigma + u(L\alpha + M\beta + N\gamma) = 0 \tag{12}$$

represents some conic passing through the S points; and we have to show that it is possible to determine L, M, N, so that (12) passes through the required six points. Combining (12) with (6), it follows that the remaining six points of intersection of (12) and (6) lie in the cubic

$$\tau(L\alpha + M\beta + N\gamma) + \alpha\beta\gamma = 0. \tag{13}$$

Since (13) passes through A, B and C, it will intersect the conic (10) in three other points; and by substituting the value of $\alpha(m\beta + n\gamma)$ from (10) in (13), one of the three chords of intersection of (10) and (13) will be found to be

$$p_1(L\alpha + M\beta + N\gamma) + \alpha = 0, \tag{14}$$

where

$$p_1 = \frac{1}{l} - \frac{1}{mn}$$

with similar expressions for p_2, p_3. The conditions that (14) should be identical with (11) are that

$$\frac{Lp_1+1}{k_1}=\frac{Mmp_1}{1-m^2}=\frac{Nnp_1}{1-n^2}.$$

By dealing with the two pairs of tangents from B and C in a similar manner, we shall obtain two more sets of equations, one of which is

$$\frac{Mp_2+1}{k_2}=\frac{Nnp_2}{1-n^2}=\frac{Llp_2}{1-l^2}.$$

These six equations are not, however, independent, and will be found to uniquely determine the values of L, M, N which are given by

$$L\Delta_1=\frac{1}{l}-l,\ \text{etc., etc.,}$$

$$\Delta_1=\frac{1+l^2+m^2+n^2}{lmn}-1-\frac{1}{l^2}-\frac{1}{m^2}-\frac{1}{n^2},$$

showing that the required conic is

$$\Delta_1\sigma+u\left\{\left(\frac{1}{l}-l\right)\alpha+\left(\frac{1}{m}-m\right)\beta+\left(\frac{1}{n}-n\right)\gamma\right\}=0, \tag{15}$$

which, therefore, passes through the S points.

A similar method can be employed to obtain the conic (7), in which the conic $2\alpha(m\beta+n\gamma)+\beta\gamma=0$ which passes through the points where the nodal tangents intersect the quartic, must be used in the place of (10).

3. *To find the equation of the conic passing through the six points of inflexion.*

The method of quadric inversion consists in writing for each coordinate its reciprocal; accordingly a straight line inverts into a conic circumscribing the triangle of reference and *vice versa;* a conic arbitrarily situated inverts into a trinodal quartic: whilst a cubic circumscribing the triangle of reference inverts into another cubic circumscribing the same triangle. Hence the quartic (1) inverts into the conic

$$S=\alpha^2+\beta^2+\gamma^2+2l\beta\gamma+2m\gamma\alpha+2n\alpha\beta=0 \tag{16}$$

and the six stationary tangents invert into six osculating conics which circumscribe the triangle of reference. Let $2U=dS/df$, $2V=dS/dg$, $2W=dS/dh$; then the conic

$$S+(\alpha U+\beta V+\gamma W)(\lambda\alpha+\mu\beta+\nu\gamma)=0 \tag{17}$$

represents a conic which osculates S at the point (f, g, h) on it; hence if (17) circumscribe the triangle of reference

$$\lambda U = \mu V = \nu W = -1,$$

accordingly since $(f, g, h,)$ lies in the line (λ, μ, ν), we obtain

$$\frac{f}{U} + \frac{g}{V} + \frac{h}{W} = 0, \tag{18}$$

which shows that the points of contact of the six conics lie in the cubics (18). Writing (α, β, γ) for (f, g, h) we obtain

$$U = \alpha + n\beta + m\gamma,$$
$$= \alpha p_1 + mnv_1,$$

where
$$v = \alpha/l + \beta/m + \gamma/n,$$
$$p_1 = 1 - mn/l, \text{ etc., etc.,}$$

whence
$$VW = p_2 p_3 \beta\gamma + lv(m\beta + n\gamma + mn\alpha),$$

accordingly the cubic (18) becomes

$$\alpha\beta\gamma\,(p_2p_3 + p_3p_2 + p_1p_2) + lmnv\,(\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma/l + 2\gamma\alpha/m + 2\alpha\beta/n) = 0.$$

Subtracting $lmn\,Sv$, and writing Δ_2 for $p_2p_3 + p_3p_1 + p_1p_2$ it follows that the cubic

$$\Delta_2\alpha\beta\gamma + 2lmn\left\{\left(\frac{1}{l} - l\right)\beta\gamma + \left(\frac{1}{m} - m\right)\gamma\alpha + \left(\frac{1}{n} - n\right)\alpha\beta\right\} v = 0, \tag{19}$$

circumscribes the triangle of reference and passes through the points of contact of the six osculating conics. Inverting (19), the cubic

$$\Delta_2\alpha\beta\gamma + 2lmn\varpi\left\{\left(\frac{1}{l} - l\right)\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma\right\} = 0,$$

passes through the nodes and the six points of inflexion of the quartic. Now this cubic is of the same form as the cubic (13), and we can therefore pass from it to the conic passing through the points of inflexion. Accordingly, the equation of the conic is

$$\Delta_2\sigma + 2lmn\left\{\left(\frac{1}{l} - l\right)\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma\right\} u = 0, \tag{20}$$

when
$$\Delta_2 = 3 + l^2 + m^2 + n^2 - 2\,(m^2n^2 + n^2l^2 + l^2m^2)/lmn.$$

It is worthy of note that the conic passing through the points of contact of the tangents from the nodes and the conic passing through the points of inflexion

intersect the conic $\sigma = 0$ in the same four points, viz., the S points and the points where σ is cut by the line

$$\left(\frac{1}{l} - l\right)\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma = 0.$$

4. *A cubic can be described through the six points where the stationary tangents intersect the quartic, which osculates the latter at the T points.*

To prove this theorem, we shall employ the parametric method in the form used by Mr. R. A. Roberts.* Let u, u'; v, v',; w, w' be the parameters of the nodes A, B and C; also let

$$p_1 = \frac{(u-v)(u-v')(u-w)(u-w')}{(u'-v)(u'-v')(u'-w)(u'-w')},$$

with similar expressions for p_2, p_3.

Then, if θ be the parameter of any point of inflexion, and x the parameter of the points where the corresponding stationary tangent cuts the quartic

$$\frac{(u-\theta)^3(u-x)}{(u'-\theta)^3(u'-x)} = p_1$$

with two similar equations. Whence

$$\prod_1^6 \left(\frac{u-\theta_n}{u'-\theta_n}\right)^3 \left(\frac{u-x_n}{u-x_n}\right) = p_1^6,$$

but since the six points of inflexion lie on a conic passing through the S points it follows that if s, s' be the parameters of these points, and t, t' those of the T points

$$\frac{(u-s)(u-s')}{(u'-s)(u'-s')} \prod_1^6 \left(\frac{u-\theta_n}{u'-\theta_n}\right) = p_1^2,$$

and

$$\frac{(u-s)(u-s')(u-t)(u-t')}{(u'-s)(u'-s')(u'-t)(u'-t')} = p_1,$$

whence

$$\frac{(u-t)^3(u-t')^3}{(u'-t)^3(u'-t')^3} \prod_1^6 \left(\frac{u-x_n}{u'-x_n}\right) = p_1^3,$$

which proves the proposition.

5. Let the tangents at the node A of a quartic cut the curve in D, D'; and let Q, q be the two remaining points of intersection of the line DD' with the quartic; then the points Q, q have important properties with respect to the

* Proc. Lond. Math. Soc., Vol. XVI, p. 44.

quartic which have been discussed by Mr. Westrop Roberts.* Now a trinodal quartic has a pair of Q points corresponding to each node, and all the properties which Mr. Roberts has proved for the Q points of a uninodal quartic are true for each pair of Q points of a trinodal quartic. In §194 of my book, I have shown that the equation of the line joining the points where the nodal tangents at A intersect the quartic is (since $2l$, $2m$, $2n$ are to be written for l, m, n),

$$2k_1\alpha + \beta/m + \gamma/n = 0. \tag{21}$$

Substituting the value of α from (21) in (1) we obtain

$$(\beta^2 + \gamma^2 + 2l\beta\gamma)\{(\beta/m + \gamma/n)^2 - 4k_1\beta\gamma\} = 0,$$

which shows that the equation of the lines AQ_1, Aq_1, is

$$(\beta/m + \gamma/n)^2 - 4k_1\beta\gamma = 0. \tag{22}$$

The equations of the lines BQ_2, Bq_2, CQ_3, Cq_3 can be written down by symmetry.

6. *A conic can be described which touches the six lines joining each node with its corresponding pair of Q points.*

The investigation of §192 of my book shows that if the equations of three pairs of straight lines are of the form

$$m\beta^2 + n\gamma^2 + \lambda\beta\gamma = 0,$$
$$n\gamma^2 + l\alpha^2 + \mu\gamma\alpha = 0,$$
$$l\alpha^2 + m\beta^2 + \nu\alpha\beta = 0,$$

a conic can be described touching them; and since (22) shows that the three pairs of lines in question are of this form, the theorem is proved.

7. *A conic can be described which passes through the nodes B and C, the two Q points corresponding to A, and the points of contact of the tangents from A; and its equation is*

$$k_1\alpha^2 + \alpha(\beta/m + \gamma/m) + \beta\gamma = 0. \tag{23}$$

This is a particular case of one of Mr. Roberts's theorems; but the proof leads to the equation of a conic which will be required in the next article. It may be verified as follows. From (21) and (22) we obtain

$$\alpha(\beta/m + \gamma/n) + 2\beta\gamma = 0, \tag{24}$$

* Proc. Lond. Math. Soc., Vol. XXV, p. 151.

which is the equation of the conic passing through the nodes and the two Q points corresponding to A. Eliminating α between (23) and (24) we obtain (22), which shows that (23) passes through the Q points corresponding to A. Eliminating α between (23) and the conic (10), which passes through the points of contact of the tangents from the node, we obtain the equation of these tangents which shows that (23) passes through their points of contact.

8. *To prove that a conic can be described through the six Q points and the two T points.*

The equation

$$\tau + u(L\alpha + M\beta + N\gamma) = 0,$$

represents some conic passing through the T points. And if we proceed in the same way as in §2 making use of the conic (24) and the line (21) in the place of the conic (10) and the line (11) we shall find that the values of L, M, N are given by the equations

$$Ll = Mm = Nn = -\Delta_3^{-1}$$

when $\Delta_3 = 1 + 4(l^2 + m^2 + n^2) - 8lmn - 2(l^2m^2 + n^2l^2 + m^2n^2)/lmn.$

9. In §287 of my book, I have proved that the locus of the point of intersection of two tangents at the extremities of a chord through the node of a nodobicuspidal quartic is a nodal cubic which passes through the cusps of the quartic. A similar proposition is true in the case of a chord drawn through a cusp, but as the cusps of a limaçon are imaginary the method is not applicable. Let the equation of the quartic be

$$(\beta\gamma + \gamma\alpha + \alpha\beta)^2 = m^2\alpha^2\beta\gamma,$$

to which form any nodobicuspidal quartic can be reduced by projection. Now it is known that any point in the quartic can be expressed in terms of a parameter θ by means of the equation

$$\alpha = -\theta^2, \quad \beta = \theta^2 - m\theta + 1, \quad \gamma = \theta^2(\theta^2 - m\theta + 1). \tag{25}$$

The polar cubic of any point (ξ, η, ζ) is

$$\xi\{2S(\beta + \gamma) - 2m^2\alpha\beta\gamma\} + \eta\{2S(\gamma + \alpha) - m^2\alpha^2\gamma\} \\ + \zeta\{2S(\alpha + \beta) - m^2\alpha^2\beta\} = 0, \tag{26}$$

where $S = \beta\gamma + \gamma\alpha + \alpha\beta$. Substituting the values of α, β, γ from (25) in (26), we obtain

$$2\theta^4(\xi + \eta) - m\theta^3(4\xi + \eta) + 2\theta^2(m^2 + 2)\zeta - m\theta(4\xi + \zeta) + 2(\xi + \zeta) = 0. \quad (27)$$

This equation determines the parameters of the points of contact of the four tangents drawn from (ξ, η, ζ).

The equation of any line through the cusp B is $\gamma + (1 - k)\alpha = 0$, and therefore the parameters of the two points where it intersects the quartic are determined by

$$\theta^2 - m\theta + k = 0. \quad (28)$$

Hence, if θ_3, θ_4 be the roots of (28), the roots of (27), must be $\theta_1, \theta_1, \theta_3, \theta_3$, and since $\theta_3 + \theta_4 = m$, $\theta_3\theta_4 = k$, we obtain from (27)

$$\theta_1 + \theta_2 + m = \tfrac{1}{2} m(4\xi + \eta)/(\xi + \eta),$$
$$m(\theta_1 + \theta_2) + \theta_1 + \theta_2 + k = (m^2 + 2)\xi/(\xi + \eta),$$
$$k(\theta_1 + \theta_2) + m\theta_1\theta_2 = \tfrac{1}{2} m(4\xi + \zeta)/(\xi + \eta),$$
$$k\theta_1\theta_2 = (\xi + \zeta)/(\xi + \eta).$$

Eliminating θ_1, θ_2 and k, we obtain

$$(m^2\eta - \zeta)\{2\xi(6\eta - m^2\eta + \zeta) + m^2\eta^2 + 2\eta\zeta\} = 18\eta^2(\xi + \zeta),$$

which is a cubic having a node at A and which passes through C.

FLEDBOROUGH HALL, HOLYPORT, BERKS, ENGLAND, *April* 21, 1903.

A Geometric Proposition.

By E. Lasker.

The elementary proposition that, "if the corners $(A, B, C)(D, E, F)$ of two triangles in a plane are such that AD, BE, CF are collinear, then the reciprocal property is true of the sides respectively opposite to the points," admits of a very wide extension that may not be without some importance. The fact, which is going to be explained, has applications to the geometry of a space of n manifoldness; but, to simplify the diction, it will be described as applying only to geometry in a plane or space. It may be announced in this fashion:

Proposition. Let A, B, C, D, E, F be six points in a plane, or A, B, C, D, E, F, G, H be eight points in space. Let Ω be any configuration of these points, which is characterized by the vanishing of one or a set of linear invariants i of above set of points respectively. In that case, any set of points forming Ω gives rise, by separating the points into two triangles or tetrahedra, to six sides or eight planes which will always again form Ω.

To be quite accurate, we add that if A, B, C, D, E, F form Ω, and the two triangles, into which the six points are divided, are $(A, B, C)(D, E, F)$, then the lines BC, CA, AB, DE, EF, FD form the reciprocal Ω. And similarly in space.

The demonstration of the proposition is thus: An invariant i linear in the coefficients of A, B, C, D, E, F is of the shape

$$c_1(ABC).(DEF) + c_2(ABD).(CEF) + c_3(ADE).(BCF) + \dots,$$

where the $c_1, c_2, c_3 \dots$ are numerical constants. Replacing in above expression A by BC, B by CA, C by AB, D by EF, E by FD, F by DE and form-

ing the reciprocal invariant i', it is found by elementary properties of determinants that

$$i' = (ABC).(DEF).i.$$

Hence, if $i = 0$, also $i' = 0$, and if a set of invariants $i = 0$, then also the corresponding set $i' = 0$. Q. E. D.

The demonstration of the proposition for higher spaces is analogous.

To give a few easy applications: If eight points A, B, C, D, E, F, G, H in space are such that the line common to the planes ABC and DEF intersects GH, then will the line common to ADE and BCF, the point of intersection of DEG and FH, and that of BCH and AG lie in one plane. Or else: If the quadric through the lines AB, CD, EF admits G, H as conjugate points, then will the quadric through CD, AB, GH admit EFH, EFG as conjugate planes; and the quadric through CG, DH and the line common to ABG and EFH will admit ABC and DEF as conjugate planes; and the quadric through CGH/DEF (read: the line common to CGH and DEF), AGH/BEF and BD will admit ACG and ACH as conjugate planes.

It is, of course, possible to apply the proposition to the linear invariants of $2m$ elements of a m-fold linear manifoldness. As an instance, let the manifoldness in question be that of the conics in a given plane whose m is 6. Let

$$u_1, u_2 \ldots\ldots u_{12}$$

be twelve conics, and let the vanishing of the invariant i signify that there is a conic belonging to all three involutions

$$(u_1, u_2, u_3, u_4),\ (u_5, u_6, u_7, u_8) \text{ and } (u_9, u_{10}, u_{11}, u_{12}).$$

Interpreting the $u_1 \ldots\ldots u_{12}$ as squares of lines, the proposition means this: "If three quadruples (l_1, l_2, l_3, l_4), $(l_5, l_6 \ldots\ldots)(l_9 \ldots\ldots)$ of lines are said to be in relation Ω whenever three conics exist touching the respective quadruples and having in common four tangents, then any system of lines $l_1 \ldots\ldots l_{12}$ in relation Ω determines systems of points $p_1 \ldots\ldots p_{12}$ also in relation Ω. The points $p_1 \ldots p_{12}$ are found in this fashion: Any two quadruples of lines $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\lambda_5 \ldots\ldots \lambda_8)$ always uniquely determine a quadruple of points $(\pi_1, \pi_2, \pi_3, \pi_4)$, namely, those four points whose squares form the linear involution comprising

the systems of conics that touch the quadruples of lines respectively. Thus,

	(p_1, p_2, p_3, p_4)
are determined by	(l_5, l_6, l_9, l_{10})
and	$(l_7, l_8, l_{11}, l_{12})$,
p_5, p_6, p_7, p_8 by	$(l_9, l_{10}, l_{11}, l_2)$
and	$(l_{11}, l_{12}, l_3, l_4)$,
$p_9, p_{10}, p_{11}, p_{12}$ by	(l_1, l_2, l_5, l_6)
and	(l_3, l_4, l_7, l_8).

These applications will have given a sufficient insight into the extent as well as the limitations of the proposition.

April, 1903.

Congruences of Tangents to a Surface and Derived Congruences.

By L. P. Eisenhart.

Given any family of curves upon a surface S; the tangents to these curves form a rectilinear congruence for which S is one of the focal sheets. The other focal sheet is a determinate surface S_1, and upon it there is a family of curves to which the lines of the congruence are tangent. If tangents are drawn to the curves on S_1, which are the conjugates of the above family, a second congruence is formed with S_1 for one of the focal sheets, and a third surface S_2 for the other. If none of these successive surfaces reduces to a curve, we get in this manner an endless sequence of congruences and at the same time an infinite suite of surfaces with a known conjugate system. In like manner the tangents to the curves on S whose directions are conjugate to the given curves form a congruence and give rise to a surface S_{-1}, so that the sequence extends in both directions. These are the *derived congruences of Darboux.**

In §1 we consider a surface S referred to a conjugate system of lines and show the relations which hold between the rectangular coordinates of a point on this surface and those of the corresponding points on the surfaces S_1 and S_{-1}. From these and similar ones for S_1 and S_{-1} can be derived those for the consecutive surfaces, and so forth. But we are not so much concerned with the general discussion of derived congruences as with the determination of those sequences of which all the congruences are particular congruences of the same kind.

Having found, in the second section, the conditions which must be satisfied in order that the conjugate system on S and S_1 be orthogonal and also on S and S_{-1}, we find that in no case can there be an infinite sequence composed entirely of *congruences of Guichard*, to use the nomenclature of Bianchi. From

* Leçons, Vol. II, p. 16 et seq.

this it follows almost immediately that there cannot exist an infinite suite of normal congruences.

In §3 are determined the conditions which the coefficients of the first fundamental form of S, referred to a conjugate system, must satisfy in order that the tangents to the curves in one family form a congruence of Ribaucour. When a similar determination is made with regard to S_1, it is found that the conditions for this surface reduce to those for S, so that whenever the tangents to the curves in both systems on S form congruences of Ribaucour, all the congruences of the suite are of this kind. These conditions are reduced considerably in form when the conjugate system on S is orthogonal. And the only isothermic surfaces satisfying these conditions are those with the linear elements

$$ds^2 = UV(du^2 + dv^2), \quad ds^2 = e^{UV}(du^2 + dv^2),$$

where U is a function of u alone and V of v alone.

The determination of these surfaces is made in §§4, 5, and in each case it is shown that all the surfaces of this class are developable.

In §6 we consider the case where the tangents to a family of curves on S form a cyclic congruence and find the conditions which the coefficients of the first quadratic form must satisfy, when the surface is referred to these curves and their conjugates. It is found that, when these curves are the lines of curvature, the congruence is also a congruence of Ribaucour. When the congruence is at the same time cyclic and of Ribaucour, there is an infinity of cyclic systems with the lines of the congruence for axes of the circles, and only in this case.

In the last section, cyclic systems of equal circles are considered, and it is shown that there cannot exist an infinite sequence of cyclic congruences for which the corresponding circles are of this kind. Incidentally, we are led to a consideration of congruences for which the developables in one system are cylindrical, and with this the discussion closes.

§1.—*General Formulae.*

Consider a surface S referred to any conjugate system of lines, $u = \text{const.}$, $v = \text{const.}$; then the rectangular coordinates, x, y, z, of its points are particular integrals of an equation of Laplace of the form

$$\frac{\partial^2\theta}{\partial u\,\partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} = 0, \tag{1}$$

where a and b are functions of u and v, whose forms are determined when the linear element

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2 \tag{2}$$

is given.

The tangents to the curves $v =$ const. form a congruence, C, for which S is one of the focal sheets. Let x_1, y_1, z_1 denote the coordinates of the point on the second sheet, S_1, which corresponds to the point (x, y, z) on S. From this definition it follows that

$$x_1 = x + \lambda \frac{\partial x}{\partial u}, \quad y_1 = y + \lambda \frac{\partial y}{\partial u}, \quad z_1 = z + \lambda \frac{\partial z}{\partial u}, \tag{3}$$

where λ is a function of u and v which is determinate and whose form will be found in a moment. Since S_1 is the second focal sheet of C, the lines of the congruence are tangent to the curves $u =$ const. on S_1, and, consequently, there exists a function μ such that

$$\frac{\partial x_1}{\partial v} = \mu \frac{\partial x}{\partial u}, \quad \frac{\partial y_1}{\partial v} = \mu \frac{\partial y}{\partial u}, \quad \frac{\partial z_1}{\partial v} = \mu \frac{\partial z}{\partial u}, \tag{4}$$

Differentiate the above expression (3) of x_1 with respect to v and equate it to the above (4); this gives, in consequence of (1),

$$\left(\mu - \frac{\partial \lambda}{\partial v} + a\lambda\right)\frac{\partial x}{\partial u} + (\lambda b - 1)\frac{\partial x}{\partial v} = 0.$$

Since this equation must be satisfied by y and z also, it follows that λ and μ must satisfy the equations

$$\mu - \frac{\partial \lambda}{\partial v} + a\lambda = 0, \quad \lambda b - 1 = 0. \tag{5}$$

Then (3) and (4) become

$$\left.\begin{aligned} x_1 &= x + \frac{1}{b}\frac{\partial x}{\partial u}, \\ \frac{\partial x_1}{\partial v} &= -\frac{1}{b^2}\left(\frac{\partial b}{\partial v} + ab\right)\frac{\partial x}{\partial u}, \end{aligned}\right\} \tag{6}$$

and similarly for y_1 and z_1. When S is given, b can be found directly, and, consequently, S_1 can be determined at once and uniquely.

In a similar manner, the tangents to the curves $u =$ const. on S form a congruence C_{-1} with S for one of the focal sheets and a new surface S_{-1} for the other focal sheet. By analogy, we have that the rectangular coordinates x_{-1}, y_{-1}, z_{-1} are given by the equations

$$\left.\begin{aligned} x_{-1} &= x + \frac{1}{a}\,\frac{\partial x}{\partial v}, \\ \frac{\partial x_{-1}}{\partial u} &= -\frac{1}{a^2}\left(\frac{\partial a}{\partial u} + ab\right)\frac{\partial x}{\partial v}, \end{aligned}\right\} \tag{7}$$

and similarly for y_{-1} and z_{-1}.

The general expression for (6) is

$$\theta_1 = \theta + \frac{1}{b}\,\frac{\partial \theta}{\partial u}, \quad \frac{\partial \theta_1}{\partial v} = -\frac{k}{b^2}\,\frac{\partial \theta}{\partial u}, \tag{8}$$

where k denotes the second invariant of the equation (1). Combining these two equations, we have

$$\theta k = \theta_1 k + b\,\frac{\partial \theta_1}{\partial v}.$$

From this it follows that, when k is zero, θ_1 is a function of u alone; similarly, when the first invariant h vanishes. Hence the theorem :*

When k vanishes, S_1 is a curve; and when h vanishes, S_{-1} is a curve.

Equation (1) can be written in the form

$$\frac{\partial}{\partial v}\left(\frac{\partial \theta}{\partial u} + b\theta\right) + a\left(\frac{\partial \theta}{\partial u} + b\theta\right) = k\theta,$$

and by (8),

$$\frac{\partial}{\partial v}(b\theta_1) + ab\theta_1 = k\theta.$$

Eliminating θ between this equation and (8), we find the following equation of which x_1, y_1, z_1 are particular solutions,

$$\frac{\partial^2 \theta_1}{\partial u\,\partial v} + a_1\frac{\partial \theta_1}{\partial u} + b_1\frac{\partial \theta_1}{\partial v} = 0,$$

where

$$a_1 = k/b, \quad b_1 = b - \frac{\partial}{\partial u}\log k/b. \tag{9}$$

* Darboux, Leçons, Vol. II, p. 21.

Similarly, the coordinates x_{-1}, y_{-1}, z_{-1} of S_{-1} satisfy the equation

$$\frac{\partial^2\theta}{\partial u\,\partial v}+a_{-1}\frac{\partial\theta}{\partial u}+b_{-1}\frac{\partial\theta}{\partial v}=0,$$

where

$$a_{-1}=\frac{h}{a},\quad b_{-1}=a-\frac{\partial}{\partial v}\log h/a. \tag{10}$$

Proceeding in this manner step by step, the equations corresponding to all the surfaces of the suite of congruences can be found and the coefficients expressed in terms of a, b and their derivatives. Without developing any further the general subject of derived congruences, we pass to a consideration of the more important particular kinds of congruences and the congruences derived from them by the preceding methods.

§ 2.—*Congruences of Guichard. Normal Congruences.*

For the congruence C to be a congruence of Guichard, it is necessary and sufficient that the conjugate systems $u=\text{const.}$, $v=\text{const.}$ on S and S_1 be orthogonal. Let S be referred to its lines of curvature and denote by α_1, β_1, γ_1 and α_2, β_2, γ_2, the direction cosines of the tangents to the curves $v=\text{const.}$, $u=\text{const.}$ respectively, and by X, Y, Z, the direction cosines of the normal to S. Between these functions the following relations hold:*

$$\frac{\partial\alpha_1}{\partial u}=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\alpha_2+\frac{D}{\sqrt{E}}X,\quad \frac{\partial\alpha_1}{\partial v}=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha_2,$$
$$\frac{\partial\alpha_2}{\partial u}=\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\alpha_1,\quad \frac{\partial\alpha_2}{\partial v}=-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha_1+\frac{D''}{\sqrt{G}}X,$$

and similarly for β_1, γ_1, β_2, γ_2, where, as usual,

$$D=\Sigma X\frac{\partial^2x}{\partial u^2},\quad D''=\Sigma X\frac{\partial^2x}{\partial v^2}.$$

Denoting by 2ρ the focal distance for the congruence, we have

$$x_1=x+2\rho\alpha_1,\quad y_1=y+2\rho\beta_1,\quad z_1=z+2\rho\gamma_1.$$

Differentiating these expressions with respect to u and v respectively, and

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* Bianchi, Lezioni, p. 94.

making use of the above relations, we get

$$\frac{\partial x_1}{\partial u} = \left(\sqrt{E} + 2\frac{\partial \rho}{\partial u}\right)\alpha_1 + 2\rho\left(-\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\alpha_2 + \frac{D}{\sqrt{E}}X\right),$$

$$\frac{\partial x_1}{\partial v} = 2\frac{\partial \rho}{\partial v}\alpha_1 + \alpha_2\left(\sqrt{G} + \frac{2\rho}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\right),$$

and similarly for y_1 and z_1. Comparing the second of these equations with (4), we see that we must have

$$\sqrt{G} + \frac{2\rho}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u} = 0. \tag{11}$$

Again, if the parametric lines on S_1 are to be orthogonal, it is necessary that

$$2\frac{\partial \rho}{\partial u} + \sqrt{E} = 0. \tag{12}$$

Differentiating (11) with respect to u, we find, in consequence of (12),

$$\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\right] = 0. \tag{13}$$

In a similar manner we find that the necessary and sufficient condition that the tangents to the lines of curvature $u = \text{const.}$ shall form a congruence of Guichard is

$$\frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\right] = 0. \tag{14}$$

When a surface is referred to its lines of curvature, the Gauss equation reduces to*

$$K = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\right]\right\}, \tag{15}$$

where K denotes the total curvature. From this it follows that for the conditions (13) and (14) to be satisfied simultaneously, it is necessary that S be a developable surface. But when S is a developable surface, one family of lines of curvature is composed of the rectilinear generatrices and hence the corresponding congruence does not exist. Hence the theorem :

In no case do the tangents to the lines of curvature in each system on a surface form congruences of Guichard.

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* Bianchi, l. c., p. 67.

And, as a consequence, the theorem:

There does not exist a series of derived congruences all of which are congruences of Guichard.

Bianchi has shown* that when S is one of the focal sheets of a congruence of Guichard, or, as he calls it, a surface of Guichard, one of the sheets of its evolute is a Voss surface, and, moreover, that the conjugate geodesics on the latter correspond to the lines of curvature upon the former. In consequence of the preceding theorems, we have that in no case are both the sheets of the evolute of a surface of Guichard surfaces of Voss.

Let S be a surface of Voss and let the parametric curves be the conjugate system of geodesics. The tangents to the latter form a normal congruence and, as Bianchi has shown,† all the surfaces cutting these lines orthogonally, are surfaces of Guichard. Since the congruence is normal, the curves $u =$ const. on S_1 are geodesics, but the curves $v =$ const. are not geodesics in consequence of the above results. Thus C and C_{-1} are normal congruences, and C_1 is not normal. Hence the theorem:

There cannot exist a derived suite of normal congruences; and, for two consecutive congruences to be normal, the common focal sheet must be a surface of Voss.

§3.—*Congruences of Ribaucour.*

From (6) we have that, if S is the first focal sheet of a congruence, the second sheet S_1 is given by

$$x_1 = x + \frac{1}{b}\frac{\partial x}{\partial u}, \quad y_1 = y + \frac{1}{b}\frac{\partial y}{\partial u}, \quad z_1 = z + \frac{1}{b}\frac{\partial z}{\partial u}. \tag{16}$$

From this it follows that the coordinates, $\bar{x}, \bar{y}, \bar{z}$, of the mean point of the line have the expressions

$$\bar{x} = x + \frac{1}{2b}\frac{\partial x}{\partial u}, \quad \bar{y} = y + \frac{1}{2b}\frac{\partial y}{\partial u}, \quad \bar{z} = z + \frac{1}{2b}\frac{\partial z}{\partial u}. \tag{17}$$

From this we get by differentiation with respect to u and v, and ready reductions by means of these equations themselves and (1), the following:

$$\frac{\partial^2 \bar{x}}{\partial u \partial v} + \left(a + \frac{1}{b}\frac{\partial b}{\partial v}\right)\frac{\partial \bar{x}}{\partial u} + b\frac{\partial \bar{x}}{\partial v} = -\frac{1}{2b}\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v}\right)\frac{\partial x}{\partial u}, \tag{18}$$

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* Loc. cit., p. 271. † Ib.

and similarly in $\bar{y}$ and $\bar{z}$. The *congruences of Ribaucour* may be defined as those for which the developables meet the mean surface of the congruence in a conjugate system. From the above discussion it is clear that the ruled surfaces $u = \text{const}$, $v = \text{const.}$ are the developables, and, consequently, it follows from (18) that the necessary and sufficient condition that the tangents to the curves $v = \text{const.}$ on S form a congruence of Ribaucour is that the functions a and b satisfy the condition

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v} = 0 . \tag{19}$$

In a similar manner we find that for the tangents to the curves $u = \text{const.}$ to form a congruence of this kind, it is necessary and sufficient that

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} - \frac{\partial^2 \log a}{\partial u \partial v} = 0 . \tag{20}$$

Subtracting these two equations of condition we get

$$\frac{\partial^2 \log ab}{\partial u \partial v} = 0 ,$$

so that for the tangents to the curves of both families to be congruences of Ribaucour it is necessary that

$$ab = UV ,$$

where U is a function of u alone and V of v alone. However, this is not the sufficient condition. Solving for a and substituting in (19), we have

$$\frac{\partial^2 \log b}{\partial u \partial v} - \frac{\partial b}{\partial v} - \frac{UV}{b^2}\frac{\partial b}{\partial u} + \frac{U'V}{b} = 0 , \tag{21}$$

where the prime denotes differentiation.

It is well known that any three independent solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} = 0 ,$$

give the rectangular coordinates of a surface S upon which the curves $u = \text{const.}$, $v = \text{const.}$ form a conjugate system. If now b is chosen arbitrarily and a is determined by quadrature from (19), every surface given by solutions of the above equation will be the focal surface of a congruence of Ribaucour. And, if

b is chosen so as to satisfy (21), each surface will be a focal sheet of two congruences of this kind.

When the condition (19) is satisfied, the point equation of the mean surface of C becomes

$$\frac{\partial^2\theta}{\partial u\partial v}+\left(a+\frac{\partial \log b}{\partial v}\right)\frac{\partial\theta}{\partial u}+b\frac{\partial\theta}{\partial v}=0.$$

Moreover, this equation has equal invariants in consequence of (19) so that we have the theorem:

The conjugate system in which a congruence of Ribaucour cuts the mean surface has equal invariants.

The condition that the point equation of S may have equal invariants is

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v}.$$

Hence, if the tangents to the curves $v=$ const. are to form a congruence of Ribaucour, we must have in consequence of (19),

$$b=UV,$$

where U and V are arbitrary functions of u and v respectively. Now

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v}=UV',$$

so that

$$a=V'\int U du+V_1.$$

If, in particular, V_1 is zero, condition (20) also is satisfied. Hence, for all surfaces whose rectangular coordinates satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\,\partial v}+UV'\frac{\partial\theta}{\partial u}+U'V\frac{\partial\theta}{\partial u}=0,$$

the tangents to the curves $v=$ const. and to the curves $u=$ const. form congruences of Ribaucour. When, in particular, U and V are constant, the surfaces S of this class are the so-called *surfaces of translation.*

The necessary and sufficient condition that the tangents to the curves $u=$ const. on S_1 form a congruence of Ribaucour is

$$\frac{\partial a_1}{\partial u}-\frac{\partial b_1}{\partial v}-\frac{\partial^2 \log a_1}{\partial u\,\partial v}=0.$$

When a_1 and b_1 are replaced by their expressions from (9), this condition reduces to (19), as was to have been expected. Again, the condition that the tangents to the curves $v =$ const. on S_1 form a congruence of Ribaucour is

$$\frac{\partial a_1}{\partial u} - \frac{\partial b_1}{\partial v} + \frac{\partial^2 \log b_1}{\partial u \partial v} = 0,$$

which can be reduced by means of (9) to

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v} + \frac{\partial^2 \log}{\partial u \partial v}\left(\frac{k}{b}\right) + \frac{\partial^2}{\partial u \partial v} \log\left[b - \frac{\partial}{\partial u} \log \frac{k}{b}\right] = 0.$$

If the condition (19) is satisfied, this equation can be reduced to (20). On account of the symmetry in all this work, we have the theorem:

Whenever the tangents to a system of lines upon a surface form a congruence of Ribaucour, and also the tangents to the conjugate curves form such a congruence, all of the derived congruences are congruences of Ribaucour.

The expressions for a and b in terms of the coefficients of the linear element of S are*

$$a = \frac{F\dfrac{\partial G}{\partial u} - G\dfrac{\partial E}{\partial v}}{2(EG - F^2)}, \quad b = \frac{F\dfrac{\partial E}{\partial v} - E\dfrac{\partial G}{\partial u}}{2(EG - F^2)}, \tag{22}$$

so that when the conjugate system on S is composed of the lines of curvature, a and b have the expressions

$$a = -\frac{\partial \log \sqrt{E}}{\partial v}, \quad b = -\frac{\partial \log \sqrt{G}}{\partial u}. \tag{23}$$

From these forms and (19) it follows that the necessary and sufficient condition that the tangents to the lines of curvature $v =$ const. on S form a congruence of Ribaucour, when S is an isothermic surface, is

$$\frac{\partial^2 \log b}{\partial u \partial v} = 0,$$

whence it follows that the linear element can be reduced, by a suitable choice of parameters, to the form

$$ds^2 = e^{UV} V_1 (du^2 + dv^2), \tag{24}$$

* Bianchi, l. c., p. 88.

where U is a function of u alone, and V and V_1 of v alone. In a similar manner it can be shown that the necessary and sufficient condition that S be an isothermic surface with the tangents to the lines of curvature $u =$ const. forming a congruence of Ribaucour, is that the linear element be reducible to the form

$$ds^2 = e^{UV} U_1 (du^2 + dv^2), \tag{25}$$

when the lines of curvature are parametric. Combining the above results, we have the theorem:

The necessary and sufficient condition that a surface be isothermic and the congruences formed by the tangents to the lines of curvature in each system be congruences of Ribaucour is that the linear element be reducible to either of the forms

$$ds^2 = UV(du^2 + dv^2), \tag{26}$$

$$ds^2 = e^{UV}(du^2 + dv^2). \tag{27}$$

When the linear element takes the first form, it follows from (23) that

$$a = -\tfrac{1}{2}\frac{V'}{V}, \quad b = -\tfrac{1}{2}\frac{U'}{U}.$$

From (5) we remark that the focal distance for the congruence C is infinite, if V is constant; and if U is a constant, the surface S_{-1} is at infinity. Similar results follow for the case where the linear element has the form (27).

Consider for a moment the case where V is constant; then both the above linear elements reduce to the form

$$ds^2 = U(du^2 + dv^2).$$

Then S is a surface of revolution and $v =$ const. are the meridians. In consequence of the preceding discussion, we have the theorem:

The tangents to the meridians of a surface of revolution form a normal congruence of Ribaucour.

We proceed now to the determination of the surfaces with the linear elements (26) and (27).

§4.—*Surfaces with the linear element* $ds^2 = UV(du^2 + dv^2)$.

Let S be a surface with the linear element (26) and the lines of curvature parametric. In order to find the surfaces of this kind, we make use of the

methods followed by Bonnet in his celebrated memoir *Sur les surfaces applicables.**

Put
$$\frac{\sqrt{E}}{\rho_{gv}} = M, \quad \frac{\sqrt{G}}{\rho_{gu}} = N, \quad \frac{\sqrt{E}}{\rho_1} = P, \quad \frac{\sqrt{G}}{\rho_2} = Q, \tag{28}$$

where ρ_{gv}, ρ_{gu} are the radii of geodesic curvature of the lines $v = \text{const.}$, $u = \text{const.}$ respectively, and ρ_1, ρ_2 are the principal radii of normal curvature corresponding to these respective directions. Bonnet shows that the above functions must satisfy the equations

$$\left.\begin{aligned} \frac{\partial P}{\partial v} + MQ = 0, \quad \frac{\partial Q}{\partial u} - NP = 0, \\ \frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \end{aligned}\right\} \tag{29}$$

and conversely that there exists a unique surface corresponding to each set of functions satisfying these equations. By means of the third the first two can be replaced by

$$\frac{\partial P^2}{\partial v} = -2PMQ = -2M\frac{\partial M}{\partial v} + 2M\frac{\partial N}{\partial u},$$
$$\frac{\partial Q^2}{\partial u} = \quad 2NPQ = -2N\frac{\partial N}{\partial u} + 2N\frac{\partial M}{\partial v},$$

or

$$\left.\begin{aligned} \frac{\partial}{\partial v}(P^2 + M^2) = 2M\frac{\partial N}{\partial u}, \\ \frac{\partial}{\partial u}(Q^2 + N^2) = 2N\frac{\partial M}{\partial v}. \end{aligned}\right\} \tag{30}$$

By a suitable choice of parameters, the linear element (26) can be put in the form

$$ds^2 = uv\left(\frac{du^2}{U} + \frac{dv^2}{V}\right), \tag{31}$$

which is more convenient for our discussion. Now

$$M = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} = -\frac{1}{2v}\sqrt{\frac{V}{U}},$$
$$N = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} = \frac{1}{2u}\sqrt{\frac{U}{V}}.$$

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* Journal de l'École Polytechnique, XLII Cahier, p. 44 et seq.

The equations (30) take the form

$$\frac{\partial}{\partial v}(P^2+M^2)=-\frac{1}{2uvU}\left(\frac{U'}{2}-\frac{U}{u}\right),$$
$$\frac{\partial}{\partial u}(Q^2+N^2)=-\frac{1}{2uvV}\left(\frac{V'}{2}-\frac{V}{v}\right),$$

from which it follows that

$$\left.\begin{aligned}P^2&=\tfrac{1}{2}\frac{1}{uU}\left(\frac{U}{u}-\frac{U'}{2}\right)\log v-\frac{V}{4v^2U}+\frac{U_2}{4U},\\ Q^2&=\tfrac{1}{2}\frac{1}{vV}\left(\frac{V}{v}-\frac{V'}{2}\right)\log u-\frac{U}{4u^2V}+\frac{V_2}{4V},\end{aligned}\right\} \tag{32}$$

where U_2 and V_2 are functions of u and v respectively, whose forms must be determined. If we square the last of equations (29) and substitute the above expressions for M, N, P, Q, this equation becomes

$$\left(\frac{V'}{v}-\frac{2V}{v^2}+\frac{U'}{u}-\frac{2U}{u^2}\right)^2=\left[\left(\frac{2U}{u^2}-\frac{U'}{u}\right)\log v-\frac{V}{v^2}+U_2\right]\left[\left(\frac{2V}{v^2}-\frac{V'}{v}\right)\log u-\frac{U}{u^2}+V_2\right].$$

If we put
$$\frac{V}{v^2}=V_1,\quad \frac{U}{u^2}=U_1, \tag{33}$$

the above equation can be written in the form

$$(vV_1'+uU_1')^2=(uU_1'\log v+V_1-U_2)(vV_1'\log u+U_1-V_2). \tag{34}$$

We consider first the case where one of the factors on the right, say the former, is equal to zero. From its form we see that we must have

$$uU_1'=a,\quad U_2=b,\quad V_1=b-a\log v,$$

where a and b are arbitrary constants. Moreover, for these values the left-hand member vanishes. Hence a solution is given by

$$U_1=\beta+a\log u,\quad V_1=b-a\log v,\quad U_2=b,\quad V_2\text{ arbitrary}. \tag{35}$$

But in this case P is zero, and hence from (28) we see that S is a developable surface. Similar results follow if the second factor vanishes.

Suppose now that neither of the right-hand factors vanish, and for the sake of brevity put

$$\theta = vV_1' + uU_1', \quad A = uU_1' \log v + V_1 - U_2, \quad B = vV_1' \log u + U_1 - V_2; \tag{36}$$

then equation (34) may be written

$$\frac{\theta^2}{A} = B. \tag{37}$$

Differentiating with respect to u and making use of the notation

$$p, q, r, t = \frac{\partial\theta}{\partial u}, \quad \frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial u^2}, \quad \frac{\partial^2\theta}{\partial v^2},$$

we get

$$\frac{2\theta p}{A} - \frac{\theta^2 (p \log v - U_2')}{A^2} = \frac{\theta}{u}.$$

As we have excluded the case where $\theta = 0$, this may be written

$$\frac{A^2}{\theta u} - 2\frac{pA}{\theta} + p \log v = U_2',$$

which, upon differentiation with respect to v, becomes

$$A^2 \frac{q}{u} - 2A\left(\frac{\theta^2}{uv} + pq\right) + \frac{p\theta^2}{v} = 0.$$

From this equation we have

$$A\frac{q}{u} = \frac{\theta^2}{uv} + pq \pm \sqrt{\frac{\theta^4}{u^2v^2} + \frac{pq\theta^2}{uv} + p^2q^2}, \tag{38}$$

If $p = 0$, this becomes

$$\frac{Aq}{u} = \frac{\theta^2}{uv} \pm \frac{\theta^2}{uv},$$

whence

$$A = 0, \quad q = 0, \quad \text{or} \quad Aq = \frac{2\theta^2}{v}.$$

The first case has been excluded. For the second to hold we must have

$$uU_1' = \alpha, \quad vV_1' = a,$$

where a and α are arbitrary constants, and equation (34) becomes

$$(\alpha + a)^2 = [(\alpha + a) \log v + b - U_2][(\alpha + a) \log u + \beta - V_2].$$

Put $(\alpha + a) \log v = v_1$, $(\alpha + a) \log u = u_1$, $b - U_2 = U_3$, $\beta - V_2 = V_3$, then the above equation becomes

$$(\alpha + a)^2 = [v_1 + U_3][u_1 + V_3].$$

Differentiating with respect to u_1, we get

$$v_1 + U_3 + U_3' (u_1 + V_3) = 0,$$

which evidently is impossible. Hence the second case cannot arise, unless $a = -\alpha$, and this is the first case. For the third case, equation (37) becomes

$$B = \frac{qv}{2},$$

which, upon differentiation with respect to u, gives $\theta = 0$. Hence p cannot vanish unless θ vanishes.

On account of the symmetry of the expression (38) for $\frac{Aq}{u}$, it follows that $\frac{Bp}{v}$ has the same expression, hence we can put

$$\frac{Aqv}{pu} = B,$$

Taking the derivative with respect to u, we get

$$\frac{(p \log v - U_2') \, qv}{pu} - \frac{Aqv \, (p + ur)}{p^2 u^2} = \frac{\theta}{u},$$

which may be written

$$\frac{\log v}{u} - \frac{A \, (p + ur)}{p^2 u^2} - \frac{\theta}{uvq} = \frac{U_2'}{pu}.$$

When this is differentiated with respect to v, it becomes

$$\frac{r}{p^2} + \frac{1}{pu} = \frac{t}{q^2} + \frac{1}{qv}.$$

From the expression for θ, it follows that the left-hand member of the above

equation does not involve v, nor the right-hand member u, so that each must be a constant; consequently, this equation can be replaced by the two

$$\frac{dp}{du} + \frac{p}{u} + \alpha p^2 = 0, \qquad \frac{dq}{dv} + \frac{q}{v} + \alpha q^2 = 0. \tag{39}$$

where α is constant.

For $\alpha = 0$, these equations give

$$p = \frac{2\beta}{u}, \quad q = \frac{2b}{v},$$

and

$$\theta = 2\beta \log u + \gamma + 2b \log v + c.$$

From the definition of θ we get

$$\left.\begin{aligned} U_1 &= \beta \log^2 u + \gamma \log u + \delta, \\ V_1 &= b \log^2 v + c \log v + d, \end{aligned}\right\} \tag{40}$$

where $\beta, \gamma, \ldots, d$ are constants whose values must be such that equation (34) will be satisfied. If we substitute these values in (34) and, for the sake of brevity, put

$$u_1 = \log u, \quad v_1 = \log v, \quad U_3 = d - U_2, \quad V_3 = \delta - V_2,$$

we get

$$\begin{aligned} &(2\beta u_1 + 2bv_1 + \gamma + c)^2 \\ &\quad = [2bu_1v_1 + (\gamma + c)u_1 + \beta u_1^2 + V_3][2\beta u_1v_1 + (\gamma + c)v_1 + bv_1^2 + U_3], \end{aligned} \tag{41}$$

Differentiating with respect to u_1 and v_1, and again with respect to these two parameters, we get

$$U_3'' V_3'' + 20b\beta = 0,$$

so that we may put

$$U_3'' = 2k\beta, \quad V_3'' = -\frac{10b}{k},$$

where k is a constant different from zero. From these equations we get

$$U_3 = k\beta u^2 + \lambda u + \mu, \quad V_3 = -\frac{5bv^2}{k} + \rho v + \sigma,$$

where $\lambda, \mu, \rho, \sigma$ are constants, such that when these expressions for U_3 and V_3

are substituted in (41), it will vanish identically. When this substitution is made, it is found that the coefficient of u_1^4 is $k\beta^2$ and of v_1^4 is $-\frac{5b^2}{k}$. Hence we must have $\beta = b = 0$. When these values are substituted (41), it is readily found that $\gamma + c = 0$, so that *formulae* (40) *reduce to* (35).

We consider finally the case where $\alpha \neq 0$. From (39) we find that

$$\frac{1}{pu} = \beta + \alpha \log u, \quad \frac{1}{qv} = b + \alpha \log v,$$

so that

$$\theta = \frac{1}{\alpha} \log(\beta + \alpha \log u) + \gamma + \frac{1}{\alpha} \log(b + \alpha \log v) + c,$$

and

$$\left.\begin{aligned} U_1 &= \frac{1}{\alpha^2}\left[(\beta + \alpha \log u)\log(\beta + \alpha \log u) - \alpha \log u\right] + \gamma \log u + \delta, \\ V_1 &= \frac{1}{\alpha^2}\left[(b + \alpha \log v)\log(b + \alpha \log v) - \alpha \log v\right] + c \log v + d. \end{aligned}\right\} \quad (42)$$

We substitute these values in equation (34) and, as before, put

$$u_1 = \log u, \quad v_1 = \log v, \quad U_3 = d - U_2, \quad V_3 = \delta - V_2;$$

then the equation becomes

$$\begin{aligned} &\left[\frac{1}{\alpha}\log(\beta + \alpha u_1) + \gamma + \frac{1}{\alpha}\log(b + \alpha v_1) + c\right]^2 \\ &\qquad = \left\{\left[c + \frac{1}{\alpha}\log(b + \alpha v_1)\right] u_1 + U_1 + V_3\right\} \\ &\qquad\quad \left\{\left[\gamma + \frac{1}{\alpha}\log(\beta + \alpha u_1)\right] v_1 + V_1 + U_3\right\}. \end{aligned} \quad (42')$$

Differentiate this equation with respect to u_1 and v_1; multiply by $(\beta+\alpha u_1)(b+\alpha v)$; differentiate the result twice with respect to u_1 and once with respect to v_1. This gives the equation

$$\begin{aligned} &[(b + \alpha v_1) V_3'' + \alpha V_3'][(\beta + \alpha u_1) U_3''' + 2\alpha U_3''] \\ &\qquad + \frac{2\alpha^2}{\beta + \alpha u_1}\left[c + \frac{1}{\alpha}\log(b + \alpha v_1) + \gamma + \frac{1}{\alpha}\log(\beta + \alpha u_1)\right] \\ &\qquad + \frac{6\alpha}{\beta + \alpha u_1} = 0. \end{aligned} \quad (43)$$

If this equation be differentiated with respect to v_1, it becomes

$$[(b+\alpha v_1)\,V_3'''+2\alpha V_3''][(\beta+\alpha u_1)\,U_3'''+2\alpha U_3'']+\frac{2\alpha^2}{(\beta+\alpha u_1)(b+\alpha v_1)}=0.$$

In accordance with this equation, we put

$$\left.\begin{aligned}(\beta+\alpha u_1)\,U_3''''+2\alpha U_3''' &= \frac{\alpha k}{\beta+\alpha u_1},\\ (b+\alpha v_1)\,V_3''''+2\alpha V_3''' &= -\frac{2\alpha}{k(b+\alpha v_1)},\end{aligned}\right\} \qquad (44)$$

where k is a constant different from zero. From the second of these equations we have by integration

$$(b+\alpha v_1)\,V_3'''+\alpha V_3''=-\frac{2}{k}\log(b+\alpha v_1)+\lambda.$$

When this expression and the first of (44) are substituted in (43), the latter becomes

$$\frac{2\alpha}{\beta+\alpha u_1}\left[\log(\beta+\alpha u_1)+\frac{k\lambda}{2}+3-\alpha(\gamma+c)\right]=0.$$

From this it follows that $\alpha=0$. Hence *the formulae* (35) *give the only solution of the problem.* We shall consider this case for a moment.

From (33) and (35) we have

$$U=u^2(\beta+\alpha\log u),\quad V=v^2(b-\alpha\log v),$$

and from (32),

$$P=0,\quad Q^2=\frac{V_2-b}{4V},$$

so that Q is an arbitrary function of v.

The linear element takes the form

$$ds^2=uv\left[\frac{du^2}{u^2(\beta+\alpha\log u)}+\frac{dv^2}{v^2(b-\alpha\log v)}\right].$$

By a suitable choice of parameters, this can be changed to

$$ds^2=e^{\alpha(u^2-v^2)}(du^2+dv^2), \qquad (45)$$

and from (28) we have

$$\frac{1}{\rho_1}=0,\quad \frac{1}{\rho_2}=e^{-\frac{\alpha u^2}{2}}V,$$

where V is an arbitrary function of v. When $a=0$, the linear element becomes

$$ds^2 = du^2 + dv^2,$$

and since ρ_2 is an arbitrary function of v alone, this class comprises all the cylinders.

§5.—*Surfaces with the linear element* $ds^2 = e^{UV}(du^2 + dv^2)$.

The discussion of this case is in every way similar to the preceding, so that we shall merely indicate the steps.

The linear element will be taken in the general form

$$ds^2 = e^{uv}\left(\frac{du^2}{U} + \frac{dv^2}{V}\right).$$

The functions M, N, P, Q have the following forms:

$$M = -\frac{v}{2}\sqrt{\frac{V}{U}}, \quad P^2 = -\frac{v^2}{8U}\left(2U + uU'\right) + \frac{U_2}{4U} - \frac{v^2 V}{4U},$$

$$N = \frac{u}{2}\sqrt{\frac{U}{V}}, \quad Q^2 = -\frac{1}{4V}\left[\frac{u^2}{2}(2V + vV') + u^2 U - V_2\right].$$

Put
$$v^2 V = V_1, \quad u^2 U = U_1, \tag{46}$$

then the last of equations (29) becomes

$$\left(\frac{V_1'}{v} + \frac{U_1'}{u}\right)^2 = \left(\frac{v^2}{2}\frac{U_1'}{u} + V_1 - U_2\right)\left(\frac{u^2}{2}\frac{V_1'}{v} + U_1 - V_2\right). \tag{47}$$

For either of the factors on the right to vanish, that is, for S to be a developable surface of the class considered, we must have

$$U_1 = au^2 + \beta, \quad V_1 = -av^2 + \gamma, \quad U_2 = \gamma, \quad V_2 \text{ arbitrary}. \tag{48}$$

Excluding this case, we put

$$\theta = \frac{V_1'}{v} + \frac{U_1}{u}, \quad A = \frac{v^2}{2}\frac{U_1'}{u} + V_1 - U_2, \quad B = \frac{u^2}{2}\frac{V_1'}{v} + U_1 - V_2,$$

and then write the equation in the form

$$\frac{\theta^2}{A} = B.$$

Differentiating this equation with respect to u and v so as to eliminate U_2 and V_2, we get finally

$$Auq = \theta^2 uv + pq \pm \sqrt{\theta^4 u^2 v^2 + pquv\theta^2 + p^2 q^2}. \tag{49}$$

As before, we can show that p can vanish only when θ is zero.

Again we remark that the right-hand member of (49) must be the expression for Bvp also, so that we have

$$\frac{Auq}{vp} = B,$$

from which we get, by differentiating with respect to u and v,

$$\frac{r}{p^2} - \frac{1}{pu} = \frac{t}{q^2} - \frac{1}{qv}.$$

Since the left-hand member does not involve v and the right-hand u, this equation may be replaced by the two

$$\frac{1}{p^2}\frac{dp}{du} - \frac{1}{pu} - 2a = 0, \quad \frac{1}{q^2}\frac{dq}{dv} - \frac{1}{qv} - 2a = 0.$$

Consider first the case where a is zero. Then we get

$$\theta = 4\beta u^2 + 2\gamma + 4bv^2 + 2c,$$

and $$U_1 = \beta u^4 + \gamma u^2 + \delta, \quad V_1 = bv^4 + cv^2 + d.$$

If we substitute these expressions in (47) and replace u^2 by u_1 and v^2 by v_1, this equation will become

$$\begin{aligned}4(2\beta u_1 + \gamma + 2bv_1 + c)^2 = [&2bu_1v_1 + (\gamma + c)u_1 \\ &+ \beta u_1^2 + \delta - V_2][2\beta u_1 v_1 + (\gamma + c)v_1 + bv_1^2 + d - U_2].\end{aligned}$$

When this is compared with (41), it is seen that the left-hand member differs by the factor 4, and the right-hand member is the same. Hence we are brought to the result that when a is zero in the above equations, no new solutions are given.

Finally, when a is different from zero,

$$\theta = \frac{1}{2a}\log(\beta + au^2) + 2\gamma + \frac{1}{2a}\log(b + av^2) + 2c,$$

and
$$U_1 = \frac{1}{4a^2}\left[(au^2+\beta)\log(au^2+\beta) - au^2\right] + \gamma u^2 + \delta\,,$$
$$V_1 = \frac{1}{4a^2}\left[(av^2+b)\log(av^2+b) - av^2\right] + cv^2 + d\,.$$

When these values are substituted in (47) and $2a$ is replaced by α, u^2 by $2u_1$, v^2 by $2v_1$, 2γ by γ, and $2c$ by c, we get (42'). Hence, as in the preceding case, *the only surfaces with the given linear element are developable, satisfying the equations* (48).

For these developable surfaces we have
$$P = 0\,, \quad Q = \frac{V_2 - \beta}{4V}\,,$$
so that Q is arbitrary, and the linear element can be brought, by a suitable choice of parameters, to the form
$$ds = e^{\sqrt{au^2+\beta}\,.\,\sqrt{b-av^2}}\,(du^2 + dv^2)\,, \tag{50}$$
and the principal radii are given by
$$\frac{1}{\rho_1} = 0\,, \quad \frac{1}{\rho_2} = e^{-\frac{1}{2}\sqrt{au^2+\beta}\,.\,\sqrt{b-av^2}}\,V.$$

As before, we note that when α is zero, the above linear element becomes
$$ds^2 = du^2 + dv^2,$$
and all the surfaces are cylindrical.

We have thus seen that all the surfaces whose linear elements take either of the forms (26) and (27) are developable. But one family of the lines of curvature of a developable surface is composed of the rectilinear generatrices and hence the tangents to these curves form a ruled surface and not a congruence. Gathering together all these results, we have the theorem:

Of all the surfaces for which the tangents to the lines of curvature in both systems form congruences of Ribaucour, none are isothermic.

§6.—*Cyclic Congruences.*

Let S be referred to any conjugate system of lines, and upon the tangents to the curves $v = \text{const.}$ as axes construct circles of radius R and center $\bar{x}$, $\bar{y}$, $\bar{z}$.

Then these coordinates will be given by expressions of the form

$$\bar{x} = x + t\frac{\partial x}{\partial u}, \quad \bar{y} = y + t\frac{\partial y}{\partial u}, \quad \bar{z} = z + t\frac{\partial z}{\partial u},$$

and the coordinates of a point on the circle have the values

$$\xi = \bar{x} + R(\alpha_1 \cos\theta + \alpha_2 \sin\theta), \quad \eta = \bar{y} + R(\beta_1 \cos\theta + \beta_2 \sin\theta),$$
$$\zeta = \bar{z} + R(\gamma_1 \cos\theta + \gamma_2 \sin\theta),$$

where $\alpha_1, \beta_1, \gamma_1$ denote the direction-cosines of the line of intersection of the tangent plane to S at (x, y, z) and the plane of the circle; $\alpha_2, \beta_2, \gamma_2$ are the direction cosines of the line in the latter plane and at right angles with the tangent plane; and θ is the angle of inclination of the radius and the former line. Bianchi* shows that the necessary and sufficient condition that there may exist a family of surfaces which cut these circles orthogonally is that the following relations be satisfied:

$$R^2\left[\frac{\partial}{\partial v}\Sigma\alpha_2\frac{\partial \alpha_1}{\partial u} - \frac{\partial}{\partial u}\Sigma\alpha_2\frac{\partial \alpha_1}{\partial v}\right] + \Sigma\alpha_1\frac{\partial \bar{x}}{\partial u}\,.\,\Sigma\alpha_2\frac{\partial \bar{x}}{\partial v} - \Sigma\alpha_2\frac{\partial \bar{x}}{\partial v}$$
$$- \Sigma\alpha_1\frac{\partial \bar{x}}{\partial v}\,.\,\Sigma\alpha_2\frac{\partial \bar{x}}{\partial u} = 0.$$

$$R\left[\Sigma\alpha_2\frac{\partial \alpha_1}{\partial v}\,.\,\Sigma\alpha_2\frac{\partial \bar{x}}{\partial u} - \Sigma\alpha_2\frac{\partial \alpha_1}{\partial u}\,.\,\Sigma\alpha_2\frac{\partial \bar{x}}{\partial v} + \frac{\partial}{\partial u}\Sigma\alpha_1\frac{\partial \bar{x}}{\partial v} - \frac{\partial}{\partial v}\Sigma\alpha_1\frac{\partial \bar{x}}{\partial u}\right]$$
$$+ \Sigma\alpha_1\frac{\partial \bar{x}}{\partial u}\frac{\partial R}{\partial v} - \Sigma\alpha_1\frac{\partial \bar{x}}{\partial v}\frac{\partial R}{\partial u} = 0.$$

$$R\left[\Sigma\alpha_2\frac{\partial \alpha_1}{\partial v}\,.\,\Sigma\alpha_1\frac{\partial \bar{x}}{\partial u} - \Sigma\alpha_2\frac{\partial \alpha_1}{\partial u}\,.\,\Sigma\alpha_1\frac{\partial \bar{x}}{\partial v} + \frac{\partial}{\partial v}\Sigma\alpha_2\frac{\partial \bar{x}}{\partial u} - \frac{\partial}{\partial u}\Sigma\alpha_2\frac{\partial \bar{x}}{\partial v}\right]$$
$$- \Sigma\alpha_2\frac{\partial \bar{x}}{\partial v}\frac{\partial R}{\partial u} + \Sigma\alpha_2\frac{\partial \bar{x}}{\partial u}\frac{\partial R}{\partial v} = 0.$$

From the definition of $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$, it follows that

$$\alpha_1, \beta_1, \gamma_1 = \frac{E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u},\ E\frac{\partial y}{\partial v} - F\frac{\partial y}{\partial u},\ E\frac{\partial z}{\partial v} - F\frac{\partial z}{\partial u}}{\sqrt{E}\sqrt{EG - F^2}},$$
$$\alpha_2, \beta_2, \gamma_2 = X, Y, Z.$$

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* L. c., p. 323.

By ready calculations we find

$$\Sigma\alpha_2\frac{\partial\alpha_1}{\partial u}=-\frac{FD}{\sqrt{EH}},\quad \Sigma\alpha_2\frac{\partial\alpha_1}{\partial v}=\frac{\sqrt{E}D''}{H},\quad \Sigma\alpha_1\frac{\partial\bar{x}}{\partial u}=\frac{tHA}{\sqrt{E}},$$

$$\Sigma\alpha_1\frac{\partial\bar{x}}{\partial v}=\frac{(1-tb)\,H}{\sqrt{E}},\quad \Sigma\alpha_2\frac{\partial\bar{x}}{\partial u}=tD,\quad \Sigma\alpha_2\frac{\partial\bar{x}}{\partial v}=0,$$

where we have put, for the sake of brevity,

$$H=\sqrt{(EG-F^2)},$$

$$A=\frac{2E\dfrac{\partial F}{\partial u}-F\dfrac{\partial E}{\partial u}-E\dfrac{\partial E}{\partial v}}{2H^2}.$$

When these values are substituted in the above equations of condition, they become

(I) $[R^2b-(1-tb)\,tE]\,D=0,$

(II) $$\frac{\sqrt{E}DD''t}{H}+\frac{\partial}{\partial u}\left[\frac{(1-tb)\,H}{\sqrt{E}}\right]-\frac{\partial}{\partial v}\left[\frac{AtH}{\sqrt{E}}\right]+\frac{AtH}{\sqrt{E}}\frac{\partial\log R}{\partial v}-\frac{(1-tb)\,H}{\sqrt{E}}\frac{\partial\log R}{\partial u}=0,$$

(III) $$\left[\frac{\partial\log t}{\partial v}+\frac{F}{Et}+\frac{\partial\log\sqrt{E}}{\partial v}-\frac{\partial\log R}{\partial v}\right]D=0.$$

From the forms of (I) and (III), it is evident that they are satisfied when the lines $v=\text{const.}$ are the generatrices of a developable surface. For the present we shall exclude this case, so that the equations (I) and (III) may be replaced by the parentheses equated to zero.

When in (II) the quantity DD'' is replaced by its expression in terms of E, F, G and their derivatives, as given by the Gauss equation, it becomes

(II) $$tb\frac{\partial\log\sqrt{E}}{\partial u}+b\left(1+\frac{\partial t}{\partial u}\right)+(1-bt)\frac{\partial\log R}{\partial u}=0.$$

Solving the equation (I) for R and substituting in (II) and (III), we get

(II′) $$\frac{\partial t}{\partial u}=-t\left(2b+\frac{\partial\log E}{\partial u}-\frac{\partial}{\partial u}\log b\right),$$

(III′) $$\frac{\partial t}{\partial v}=-\frac{2F}{E}+t\left(\frac{2bF}{E}-\frac{\partial\log b}{\partial v}\right).$$

Differentiating the first with respect to v and the second with respect to u and subtracting, we get

$$\text{(IV)} \qquad t\left(\frac{\partial a}{\partial u}-\frac{\partial b}{\partial v}+\frac{\partial^2 \log b}{\partial u \partial v}\right)+\frac{F}{E}\left(2b+\frac{\partial \log F}{\partial u}-\frac{\partial}{\partial u}\log b\right)=0\,.$$

When the conjugate system on S is composed of the lines of curvature, this equation reduces to

$$\frac{\partial a}{\partial u}-\frac{\partial b}{\partial v}+\frac{\partial^2 \log b}{\partial u \partial v}=0\,. \tag{19}$$

Hence we have the theorem :

When the tangents to the lines of curvature in one system on a surface form a cyclic congruence, it is at the same time a congruence of Ribaucour ; and, conversely.

Recalling some of the results found in the study of congruences of Ribaucour, we have the theorem :

The tangents to the meridians of any surface of revolution form a normal cyclic congruence of Ribaucour.

And

There are no isothermic surfaces for which the tangents to the lines of curvature in both systems form cyclic congruences.

From (IV) we have that, when the parametric lines are not the lines of curvature, the tangents to the curves $v=$ const. form a cyclic congruence of Ribaucour if the functions E, F, G satisfy (19) and

$$2b+\frac{\partial \log F}{\partial u}-\frac{\partial}{\partial u}\log b=0\,. \tag{51}$$

Again, for the tangents to the lines $u=$ const. to form a cyclic congruence of Ribaucour, the functions (E, F, G) must satisfy the equations

$$\frac{\partial a}{\partial u}-\frac{\partial b}{\partial v}-\frac{\partial^2 \log a}{\partial u \partial v}=0\,, \tag{20}$$

$$2a+\frac{\partial \log F}{\partial v}-\frac{\partial}{\partial v}\log a=0\,. \tag{52}$$

If we differentiate (51) with respect to v and (52) with respect to v, and subtract, we get

$$2\,\frac{\partial b}{\partial v}-\frac{\partial^2 \log b}{\partial u \partial v}=2\,\frac{\partial a}{\partial u}-\frac{\partial^2 \log a}{\partial u \partial v}\,,$$

so that, if (51) and (52) are given, and either of (19) and (20), the other follows in consequence of this equation. Hence we may say that *the necessary and sufficient condition that the two congruences of tangents to the curves of a conjugate system on a surface are cyclic congruences of Ribaucour, is either that the curves be orthogonal and equations* (19) *and* (20) *hold, or that equations* (51), (52) *and either* (19) *or* (20) *be satisfied.*

Let S be referred to its lines of curvature; then from (6) we get

$$F_1 = \Sigma \frac{\partial x_1}{\partial u} \frac{\partial x_1}{\partial v} = \frac{E}{b^3}\left(\frac{\partial b}{\partial v} + ab\right)\left(\frac{\partial}{\partial u} \log b - b - \frac{\partial \log \sqrt{E}}{\partial u}\right).$$

Since the function $\left(\frac{\partial b}{\partial v} + ab\right)$ is the invariant k, it cannot be zero unless S_1 is a curve. As this case is excluded, we have that the necessary and sufficient condition that the conjugate system on S_1 be formed of the lines of curvature, and, therefore, that the tangents to the curves $v =$ const. on S form a congruence of Guichard, is

$$\frac{\partial}{\partial u} \log b - b - \frac{\partial \log \sqrt{E}}{\partial u} = 0.$$

Differentiating with respect to v and making use of (23), we get (19), which leads to the well-known theorem:

The congruences of Guichard are congruences of Ribaucour.

The above equation may be written

$$\frac{\partial}{\partial u} \log b + \frac{\partial}{\partial u} \log \sqrt{G} - \frac{\partial}{\partial u} \log \sqrt{E} = 0,$$

whence we have

$$\frac{\partial \sqrt{G}}{\partial u} = V\sqrt{E},$$

so that

$$\sqrt{G} = V\int \sqrt{E}\, du + V_1. \tag{53}$$

From the above we have that V can have a zero value only in case G is a function of v alone.

In a similar manner we can show that for F_{-1} to be zero we must have

$$\sqrt{E} = U\int\sqrt{G}\,dv + U_1. \tag{54}$$

We have shown in §1 that conditions (53) and (54) cannot be satisfied simultaneously.

When equation (19) and either $F=0$ or (51) are satisfied, t is given by quadrature from (II′) and (III′), and, consequently, involves an arbitrary constant. Hence, *when a cyclic congruence is also a congruence of Ribaucour, there is an infinity of cyclic systems whose circles have the lines of the congruence for axes.*

In order to consider the case where the cyclic congruence is not a congruence of Ribaucour, we write (IV) in the form

$$\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u\,\partial v}\right)(1-2tb) = \frac{Fb}{E}\left[\frac{\partial \log F}{\partial u} - 3\,\frac{\partial \log b}{\partial u} + \frac{\partial \log E}{\partial u} + 4b\right]$$
$$- \frac{\partial^2 \log\sqrt{E}}{\partial u \partial v} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u\,\partial v} \tag{55}$$

We denote by δ the semi-focal distance and by d the distance from the mean point to the center of the circle. Since it is a characteristic property of cyclic congruences that the focal points lie on the tangents to the lines of curvature of any surface orthogonal to the circle,* we have the relation

$$d^2 + R^2 = \delta^2,$$

hence a real angle σ exists defined by

$$\cos\sigma = \frac{d}{\delta}\,.$$

It is readily found that

$$\cos\sigma = (1 - 2tb),$$

Hence, when the ratio of the right-hand member of equation (55) to the coefficient of $1 - 2tb$ is, in absolute value, less than unity, the congruence of tangents to the curves $v=$ const. is cyclic. Since the function t corresponding to this case is given by (IV) and consequently doesn't contain any arbitrary quantity, there is only one cyclic system with the lines of the congruence for axes.

The preceding equations are of too complicated a form to enable us to solve the general problem of the derived congruences of a cyclic congruence. We

* Tzitzeica, Thesis.

will therefore close this discussion with an investigation of cyclic systems of equal circles in relation to the above problem.

§7.—*Cyclic systems of equal circles.*

Bianchi has shown that there are only two ways in which cyclic systems of equal circles can be formed; either by describing circles of radius R in the tangent planes to a pseudospherical surface of curvature $-\frac{1}{R^2}$ and with centers at the points of contact, or by drawing tangents to the geodesic lines of curvature of a surface of Monge and with points at the constant distance R from the points of contact as centers describing circles of radius R in the plane of the geodesic, and hence cutting the surface orthogonally.

For the first case the axes of the circles are the normals to the pseudospherical surface. Since these congruences are normal, there cannot be a sequence of derived congruences of this kind.

We pass now to the second method of forming a cyclic system of equal circles. Let the geodesic lines of curvature be $v =$ const. Since these lines are plane, the infinity of circles which meet the surface in points of a line $v =$ const. lie in the same plane and, consequently, their axes have the same direction; from this it follows that the direction-cosines of the lines of the congruence are functions of v alone. Hence, one of the focal sheets will be at infinity and the other will be the envelope of the cylinder, whose right-section is the locus of centers of the circles, when the plane $v =$ const. rolls without sliding upon its generator. As one of the focal sheets is at infinity, there cannot be a suite of derived congruences of this kind. Gathering together the preceding results, we have the theorem:

There cannot exist a sequence of cyclic congruences for which the circles of each congruence are equal.

Incidentally we have been brought to the following result: Given a surface S which is the envelope of a cylinder depending upon a single parameter and of invariable right-section. If in the plane of any right-section, circles of equal radius are described with points of the right section for centers, these circles generate a cyclic system as the cylinder envelopes S and the surfaces cutting the circles orthogonally are surfaces of Monge.

The analytical condition for this is readily found. Thus let S be the envelope of a cylinder and let the congruence of elements of the cylinder have the curves $v =$ const. for edges of regression. The direction-cosines of these lines are

$$\frac{1}{E}\frac{\partial x}{\partial u}, \quad \frac{1}{E}\frac{\partial y}{\partial u}, \quad \frac{1}{E}\frac{\partial z}{\partial u},$$

Since S is the envelope of the cylinder, the lines of the congruence meeting S along the conjugate directions $u =$ const. must have the same directions, so that the derivatives with respect to v of the above direction-cosines must be zero. Thus,

$$\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right) = \left(\frac{\partial}{\partial v}\frac{1}{\sqrt{E}} - \frac{a}{\sqrt{E}}\right)\frac{\partial x}{\partial u} - \frac{b}{\sqrt{E}}\frac{\partial x}{\partial v} = 0.$$

Since the same equation must be satisfied by y and z, we must have

$$\frac{\partial}{\partial v}\frac{1}{\sqrt{E}} - \frac{a}{\sqrt{E}} = 0, \quad b = 0.$$

When a and b are replaced by their expressions (22), it is found that these two equations are the same, namely,

$$E\frac{\partial G}{\partial u} - F\frac{\partial E}{\partial v} = 0. \tag{56}$$

Similarly, for the tangents to the curves $u =$ const. to form such a congruence, we must have

$$F\frac{\partial G}{\partial u} - G\frac{\partial E}{\partial v} = 0. \tag{57}$$

The necessary and sufficient condition that these two conditions be satisfied simultaneously, is that the point equation of S becomes

$$\frac{\partial^2 \theta}{\partial u \partial v} = 0.$$

Hence the theorem:

The tangents to each family of generating curves of a surface of translation form congruences for which one family of the developable surfaces is composed of cylinders.

For the conjugate system on S to be orthogonal and condition (56) be satisfied, G must be a function of v alone ; that is, the lines of curvature $u =$ const. must be geodesics, and hence S a surface of Monge. From the properties of these surfaces and a preceding remark, we have the theorem :

Surfaces of Monge are the envelopes of cylinders, of unvariable right-section, depending upon a single parameter. Moreover, if any right-section of this generating cylinder is made and with the points of the section as centers, circles of equal radius are described in the plane of the section ; these circles form a cyclic system and the orthogonal surfaces are surfaces of Monge.

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Bemerkungen zu Herrn D. N. Lehmer's Abhandlung in Bd. 22 dieses Journals, S. 293-335.

Von Edmund Landau.

In seiner Abhandlung "asymptotic evaluation of certain totient sums" formuliert Herr Lehmer am Schluss ein Problem, auf welches er durch folgende Thatsachen geführt worden ist.

Es bezeichne $\nu(n)$ die Anzahl der verschiedenen Primfactoren der ganzen Zahl n, und es möge unter $\Theta_{(4,1)}(n)$ 1 oder 0 verstanden werden, je nachdem alle Primfactoren der Zahl n von der Form $4m+1$ sind oder nicht. Dann lässt sich, wie Herr Lehmer* zeigt, aus der Theorie der binären quadratischen Formen mit der Discriminante -1 ohne erhebliche Schwierigkeit folgern, dass der Quotient

$$\frac{\sum_{n=1}^{x} 2^{\nu(n)}\Theta_{(4,1)}(n)}{x}$$

sich für $x=\infty$ einer endlichen Grenze nähert; es ergiebt sich nämlich, dass

$$\lim_{x=\infty}\frac{\sum_{n=1}^{x} 2^{\nu(n)}\Theta_{(4,1)}(n)}{x}=\frac{1}{\pi}$$

ist.

Ebenso zeigt er† mit Hilfe der Theorie der quadratischen Formen, deren Discriminante -3 ist: wenn $\Theta_{(6,1)}(n)$ 1 oder 0 bedeutet, je nachdem alle Primfac-

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* S. 328, Z. 24.

† S. 331, Z. 7., wenn $\Delta=3$ eingesetzt wird.

toren von n die Gestalt $6m+1$ haben oder nicht, so existiert der Grenzwert

$$\lim_{x=\infty} \frac{\sum_{n=1}^{x} 2^{\nu(n)} \Theta_{(6,1)}(n)}{x}$$

und ist $= \frac{\sqrt{3}}{2\pi}$.

Die Theorie der quadratischen Formen mit einer beliebigen Discriminante D liefert keinen weiteren Satz dieser Art, sondern, wie Herr Lehmer* zeigt, für jedes D einen Satz, in welchem das System der Linearformen auftritt, für welche D quadratischer Rest von den durch sie dargestellten Primzahlen ist. Es ergiebt sich also auf diesem Wege kein Satz mehr, in welchem es sich nur um eine einzige Linearform handelt.

Bereits der Fall der Progression $4m+3$ ist mit den elementaren Methoden nicht angreifbar; Herr Lehmer spricht nur die Vermutung† aus, dass für $x=\infty$ der Grenzwert des Quotienten

$$\frac{\sum_{n=1}^{x} 2^{\nu(n)} \Theta_{(4,3)}(n)}{x}$$

existiert und den Wert $\frac{2}{\pi}$ hat. Er beschliesst seine Abhandlung mit dem Desideratum: ‡

a und b seien ein Paar teilerfremder Zahlen; $\Theta_{(a,b)}(n)$ bedeute 1 oder 0, je nachdem alle $\nu(n)$ verschiedenen Primfactoren der Zahl n die Form $am+b$ haben oder nicht. Es soll die Richtigkeit oder Unrichtigkeit der Gleichung

$$\lim_{x=\infty} \frac{\sum_{n=1}^{x} 2^{\nu(n)} \Theta_{(a,b)}(n)}{x} = \text{constans}$$

nachgewiesen werden.

Der Beweis der Existenz dieses Grenzwertes würde—so fügt er hinzu—zugleich den Dirichlet'schen Satz vom Vorhandensein unendlich vieler Primzahlen in der Progression $am+b$ ergeben. Er vermutet also, dass der Grenzwert für jede Progression existiert und gleich einer *von Null verschiedenen* Con-

* S. 331, Z. 7 und 9. † S. 331, Z. 14. ‡ S. 334–335.

stanten ist. Im Gegensatz zu dieser Vermutung ist jedoch—wie im §1 des Folgenden ausgeführt werden soll—leicht beweisbar und zwar allein auf Grund der Dirichlet'schen Arbeit (d. h., ohne Anwendung neuerer Untersuchungen über die arithmetische Progression): für jede Progression $am + b$, in der $a > 6$ oder $a = 5$ ist, ist jener Grenzwert, wenn er existiert, gleich Null. Der Nachweis seiner Existenz würde also nicht das Vorhandensein unendlich vieler Primzahlen in der Progression zugleich ergeben.

Ein von Null verschiedener Grenzwert kann also nur in endlich vielen Fällen vorhanden sein, nämlich höchstens für die sechs Progressionen $3m + 1$, $3m + 2$, $4m + 1$, $4m + 3$, $6m + 1$, $6m + 5$. Von diesen stimmt in Bezug auf die vorkommenden Primzahlen die erste mit der fünften überein, während die zweite sich von der sechsten nur dadurch unterscheidet, dass sie die Primzahl 2 mehr enthält. Nach dem oben Erwähnten sind die Fälle $3m + 1$, $4m + 1$ und $6m + 1$ durch Herrn Lehmer einfach erledigt, und es bleiben die Fälle $3m + 2$, $4m + 3$ und $6m + 5$ offen. Wird die Existenz des betreffenden Grenzwertes als bewiesen angenommen, so ist seine Bestimmung ganz leicht, und es ergiebt sich insbesondere für die Progression $4m + 3$ der von Herrn Lehmer vermutete Wert $\frac{2}{\pi}$. Aber der springende Punkt besteht—wie bei vielen auf die Verteilung der Primzahlen bezüglichen Fragen—in dem Nachweise der Existenz des Grenzwertes.

Es ist mir nun gelungen, mit Hilfe der neueren analytischen Hilfsmittel, insbesondere der in meiner Arbeit* "Über die Primzahlen einer arithmetischen Progression" bewiesenen Sätze jenen Nachweis für alle drei Fälle zu führen, und es soll die Aufgabe des Folgenden (von §2 an) sein, diesen Beweis darzulegen.

§1.

Die unendliche Reihe

$$\sum_q \frac{1}{q^s},$$

in welcher q alle Primzahlen $am + b$ durchläuft, ist für $s > 1$ convergent, und,

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* Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, mathem.-naturw. Klasse, Bd. 112, Abt. 2ª, 1903, S. 493–535.

wie Dirichlet* bewiesen hat, die Differenz

$$\sum_q \frac{1}{q^s} - \frac{1}{\phi(a)} \log \frac{1}{s-1}$$

bleibt endlich,† wenn s gegen 1 abnimmt. Wird also

$$\sum_q \frac{1}{q^s} = \frac{1}{\phi(a)} \log \frac{1}{s-1} + E_1(s)$$

gesetzt, so ist $E_1(s)$ (desgl. im Folgenden $E_2(s), \ldots, E_5(s)$) eine Function, welche für alle den Ungleichungen

$$1 < s < 2‡$$

genügenden Werte von s dem absoluten Betrage nach unterhalb einer endlichen Schranke gelegen ist.

Da ferner die Summe

$$\sum_q \left(\frac{1}{2}\frac{1}{q^{2s}} + \frac{1}{3}\frac{1}{q^{3s}} + \cdots + \frac{1}{k}\frac{1}{q^{ks}} + \cdots\right)$$

endlich § bleibt, wenn s gegen 1 convergiert, so ist

$$\sum_q \log \frac{1}{1-\frac{1}{q^s}} = \sum_q \left(\frac{1}{q^s} + \frac{1}{2}\frac{1}{q^{2s}} + \frac{1}{3}\frac{1}{q^{3s}} + \cdots\right)$$

$$= \sum_q \frac{1}{q^s} + \sum_q \left(\frac{1}{2}\frac{1}{q^{2s}} + \frac{1}{3}\frac{1}{q^{3s}} + \cdots\right) = \frac{1}{\phi(a)} \log \frac{1}{s-1} + E_2(s),$$

also

$$\prod_q \frac{1}{1-\frac{1}{q^s}} = e^{\sum_q \log \frac{1}{1-\frac{1}{q^s}}} = e^{\frac{1}{\phi(a)} \log \frac{1}{s-1}} e^{E_2(s)} = \frac{1}{(s-1)^{\frac{1}{\phi(a)}}} E_3(s). \qquad (1)$$

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* "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält," Abhandlungen der Königlich Preussischen Akademie der Wissenschaften zu Berlin, 1837, S. 45–71; Werke, Bd. 1, 1889, S. 313–342.

† Dirichlet hat sogar bewiesen, dass jene Differenz sich für $s=1$ einem Grenzwerte nähert; doch ist diese Kenntnis für den vorliegenden Zweck unerheblich.

‡ Statt 2 könnte natürlich hier auch jede andere oberhalb 1 gelegene Grösse stehen.

§ Auch diese Function hat sogar für $s=1$ einen Grenzwert, nämlich $\sum_q \left(\frac{1}{2}\frac{1}{q^2} + \frac{1}{3}\frac{1}{q^3} + \cdots\right)$.

Andererseits bleibt das unendliche Produkt

$$\prod_q \left(1 - \frac{1}{q^{2s}}\right)$$

endlich,* wenn s gegen 1 convergiert:

$$\prod_q \left(1 - \frac{1}{q^{2s}}\right) = E_4(s). \tag{2}$$

Durch Quadrieren von (1) und Multiplizieren mit (2) ergiebt sich

$$\prod_q \frac{1 - \frac{1}{q^{2s}}}{\left(1 - \frac{1}{q^s}\right)^2} = \frac{1}{(s-1)^{\frac{2}{\phi(a)}}} E_3^2(s) E_4(s) = \frac{1}{(s-1)^{\frac{2}{\phi(a)}}} E_5(s). \tag{3}$$

Nun ist aber

$$\prod_q \frac{1 - \frac{1}{q^{2s}}}{\left(1 - \frac{1}{q^s}\right)^2} = \prod_q \frac{1 + \frac{1}{q^s}}{1 - \frac{1}{q^s}} = \prod_q \frac{q^s + 1}{q^s - 1} = \prod_q \left(1 + \frac{2}{q^s - 1}\right)$$

$$= \prod_q \left(1 + \frac{2}{q^s} + \frac{2}{q^{2s}} + \frac{2}{q^{3s}} + \dots\right),$$

und dies ist offenbar

$$= \sum_{n=1}^{\infty} \frac{2^{\nu(n)} \Theta_{(a,b)}(n)}{n^s};$$

denn beim Ausmultiplizieren treten im Nenner die s-ten Potenzen aller derjenigen Zahlen auf, deren Primfactoren sämtlich die Form $am + b$ haben, und zwar entspricht einem wirklich vorkommenden Nenner n^s gerade der Zähler $2^{\nu(n)}$.† Man erhält somit wegen (3)

$$\sum_{n=1}^{\infty} \frac{2^{\nu(n)} \Theta_{(a,b)}(n)}{n^s} = \frac{1}{(s-1)^{\frac{2}{\phi(a)}}} E_5(s),$$

$$(s-1) \sum_{n=1}^{\infty} \frac{2^{\nu(n)} \Theta_{(a,b)}(n)}{n^s} = (s-1)^{1 - \frac{2}{\phi(a)}} E_5(s). \tag{4}$$

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* Es nähert sich sogar für $s = 1$ dem Grenzwert $\prod_q \left(1 - \frac{1}{q^2}\right)$.

† Unter $\Theta_{(a,b)}(1)$ ist 1 zu verstehen.

Für $a > 6$ und $a = 5$ ist $\phi(a) > 2$, also

$$1 - \frac{2}{\phi(a)} > 0;$$

daher nähert sich alsdann die rechte Seite von (4) dem Grenzwerte 0, falls s zu 1 abnimmt.

Nun besagt ein bekannter Satz von Dirichlet*): Es stelle c_n für jedes ganzzahlige $n = 1, 2, 3, \ldots$ eine ganze Zahl $\geqq 0$ dar; wenn

$$\lim_{x=\infty} \frac{\sum_{n=1}^{x} c_n}{x}$$

existiert und $= \omega$ ist, so convergiert die Reihe

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

für $s > 1$, und es existiert

$$\lim_{s=1} \left((s-1) \sum_{n=1}^{\infty} \frac{c_n}{n^s} \right)$$

und ist $= \omega$.

Wenn also

$$\lim_{x=\infty} \frac{\sum_{n=1}^{x} 2^{\nu(n)} \Theta_{(a,\,b)}(n)}{x}$$

für eine Progression $am + b$, deren Differenz $a > 6$ oder $= 5$ ist, existiert und $= \omega$ ist, so kann nach (4) nur $\omega = 0$ sein, was bewiesen werden sollte.

Ein von Null verschiedener Grenzwert kann also—abgesehen von den durch Herrn Lehmer erledigten Fällen—nur für die Progressionen $3m + 2$, $4m + 3$ und $6m + 5$ vorhanden sein; ich werde im Folgenden für diese drei Fälle den Existenzbeweis erbringen und beginne mit dem Fall $4m + 3$.

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* "Sur un théorème relatif aux séries," Journal de mathématiques pures et appliquées, Ser. 2., Bd. 1, 1856, S. 80–81; Journal für die reine und angewandte Mathematik, Bd. 53, 1857, S. 130–131; Werke, Bd. 2, 1897, S. 198; Vorlesungen über Zahlentheorie, 4. Aufl., 1894, S. 306.

§ 2.

Es werde
$$2^{\nu(n)}\,\Theta_{(4,3)}(n) = c_n$$
gesetzt; s werde als complexe Variabele $\sigma + ti$ aufgefasst und $g(s)$ bezeichne die analytische Function, welche durch die für $\sigma > 1$ convergente Reihe
$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$
definiert ist. Dann ist für $\sigma > 1$
$$g(s) = \prod_q \left(1 + \frac{2}{q^s} + \frac{2}{q^{2s}} + \cdots + \frac{2}{q^{ks}} + \cdots\right)$$
$$= \prod_q \left(1 + \frac{2}{q^s - 1}\right) = \prod_q \frac{q^s + 1}{q^s - 1} = \prod_q \frac{1 + \frac{1}{q^s}}{1 - \frac{1}{q^s}}, \tag{5}$$
wo q alle Primzahlen $4m + 3$ durchläuft.

Wenn $\zeta(s)$ die Riemann'sche Zetafunction bezeichnet, so ist für $\sigma > 1$
$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$
wo p alle Primzahlen durchläuft, also
$$\zeta(s) = \frac{1}{1 - \frac{1}{2^s}} \prod_q \frac{1}{1 - \frac{1}{q^s}} \prod_r \frac{1}{1 - \frac{1}{r^s}}, \tag{6}$$
wo r alle Primzahlen $4m + 1$ durchläuft. Aus (6) folgt
$$\prod_q \frac{1}{1 - \frac{1}{q^s}} = \left(1 - \frac{1}{2^s}\right)\zeta(s) \prod_r \left(1 - \frac{1}{r^s}\right);$$
dies ergiebt, in (5) eingesetzt,
$$g(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) \prod_q \left(1 + \frac{1}{q^s}\right) \prod_r \left(1 - \frac{1}{r^s}\right). \tag{7}$$

Es sei eine zahlentheoretische Function $\chi(n)$ so definiert, dass

$$\chi(4m) = 0, \quad \chi(4m+1) = 1, \quad \chi(4m+2) = 0, \quad \chi(4m+3) = -1$$

ist; das ist der vom Hauptcharakter verschiedene Charakter der Gruppe der zu 4 teilerfremden Restklassen modulo 4. Für Primzahlen ist speziell

$$\chi(2) = 0, \quad \chi(q) = -1, \quad \chi(r) = 1,$$

und (7) verwandelt sich in

$$g(s) = \left(1 - \frac{1}{2^s}\right) \zeta(s) \prod_p \left(1 - \frac{\chi(p)}{p^s}\right).$$

Weil nun für $\sigma > 1$

$$\prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

ist, ist für $\sigma > 1$

$$g(s) = \left(1 - \frac{1}{2^s}\right) \zeta(s) \frac{1}{\sum\limits_{n=1}^{\infty} \frac{\chi(n)}{n^s}}. \tag{8}$$

(8) leistet, wenn die Grundeigenschaften der ζ-Function angewendet werden, die Fortsetzung der Function $g(s)$ über die Gerade $\sigma = 1$ hinaus; $g(s)$ ist für $\sigma > 0$* meromorph und hat in dieser Halbebene zu Polen 1) alle Nullstellen der Reihe $\sum\limits_{n=1}^{\infty} \frac{\chi(n)}{n^s}$, soweit es nicht Nullstellen mindestens gleicher Ordnung von $\zeta(s)$ sind, 2) den Punkt $s = 1$; dieser ist für $g(s)$ Pol erster Ordnung mit dem Residuum

$$(1 - \tfrac{1}{2}) \frac{1}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots} = \frac{1}{2} \cdot \frac{1}{\frac{\pi}{4}} = \frac{2}{\pi}.$$

§3.

In meiner Arbeit über die arithmetische Progression habe ich u. a. folgende beiden Sätze bewiesen:

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* Weiter braucht man für den vorliegenden Zweck nicht zu gehen.

I.* Es ist für
$$t \geqq 3,\ 1 - \frac{1}{\log t} \leqq \sigma \leqq 2$$
$$|\zeta(s)| < c_1 \log t, \tag{9}$$
wo c_1 eine Constante ist. D. h. der Quotient
$$\frac{\zeta(s)}{\log t}$$
bleibt in jenem Teil der Ebene endlich.

II.† Die Reihe
$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{10}$$
verschwindet für kein s mit reellem Teil $\geqq 1$ (dies war schon vorher bekannt), und es giebt zwei Constanten c_2 und c_3, so dass in dem Gebiete $t \geqq 3$, $1 - \frac{1}{\log^{c_2} t} \leqq \sigma \leqq 2$ die Reihe (10) nicht verschwindet und der Ungleichung
$$\left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \right| > \frac{1}{\log^{c_3} t} \tag{11}$$
genügt.

Da ferner für diese Werte von s
$$\left| 1 - \frac{1}{2^s} \right| \leqq 1 + \frac{1}{2^{1 - \frac{1}{\log^{c_2} 3}}} = c_4$$
ist, so folgt für sie in Verbindung mit (8), (9) und (11)
$$|g(s)| < c_4 \cdot c_1 \log t \cdot \log^{c_3} t < \log^{c_5} t.$$

Daher ergiebt sich der Satz, welcher genau dem Satz l. c., S. 524, entspricht: $g(s)$ bezeichne die für $\sigma > 1$ durch die Dirichlet'sche Reihe
$$\sum_{n=1}^{\infty} \frac{c_n}{n^s} = \sum_{n=1}^{\infty} \frac{2^{\nu(n)} \Theta_{(4,3)}(n)}{n^s}$$
definierte Function; dann giebt es eine positive Zahl α mit folgenden Eigenschaften: $g(s)$ ist in demjenigen Teil der Ebene eindeutig und abgesehen vom Pole erster Ordnung $s = 1$ (wo das Residuum $= \frac{2}{\pi}$ ist) regulär, welcher rechts

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* l. c., S. 514. † l. c., S. 521.

von der stetigen Curve

$$\sigma = 1 - \frac{1}{\log^a t} \text{ für } t \geqq 3,$$

$$\sigma = 1 - \frac{1}{\log^a 3} \text{ für } -3 \leqq t \leqq 3,$$

$$\sigma = 1 - \frac{1}{\log^a(-t)} \text{ für } t \leqq -3$$

liegt (incl. der Curve selbst), und $g(s)$ genügt für $t \geqq 3$, $1 - \frac{1}{\log^a t} \leqq \sigma \leqq 2$ der Ungleichung

$$|g(\sigma + ti)| = |g(\sigma - ti)| < \log^a t. \tag{12}$$

§4.

Da für $\sigma = 2$ die Reihe

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

gleichmässig convergiert, so ergiebt sich genau wie in §9 * der Arbeit über die arithmetische Progression

$$\sum_{n=1}^{x} c_n \log \frac{x}{n} = \frac{1}{2\pi i} \int_{2-x^2 i}^{2+x^2 i} \frac{x^s}{s^2} g(s)\, ds + O(1), \tag{13}$$

wo das Integral geradlinig zu erstrecken ist.

§5.

Wörtlich wie a. a. O. in §10† mit Hilfe der dort mit (33) bezeichneten Ungleichung ergiebt sich hier mit Hilfe von (12), dass das in (13) auftretende Integral

$$\int_{2-x^2 i}^{2+x^2 i} \frac{x^s}{s^2} g(s)\, ds = 2\pi i \,.\, \frac{2}{\pi} x + O(xe^{-\sqrt[c]{\log x}}) \tag{14}$$

ist, wo c eine positive Constante bezeichnet.

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* L. c., S. 525–526. † L. c., S. 526–529.

Aus (14) folgt in Verbindung mit (13), dass

$$\sum_{n=1}^{x} c_n \log \frac{x}{n} = \frac{2}{\pi} x + O(xe^{-\sqrt[c]{\log x}}) \tag{15}$$

ist.

§6.

Aus (15) folgt, wenn

$$e^{-\sqrt[2c]{\log x}} = \delta$$

gesetzt wird,

$$\sum_{n=1}^{(1+\delta)x} c_n \log \frac{(1+\delta)x}{n} = \frac{2}{\pi}(x + \delta x) + O(xe^{-\sqrt[c]{\log x}}).* \tag{16}$$

Durch Subtraction erhält man aus (15) und (16)

$$\log(1+\delta) \sum_{n=1}^{x} c_n + \sum_{n=x+1}^{(1+\delta)x} c_n \log \frac{(1+\delta)x}{n} = \frac{2}{\pi}\delta x + O(xe^{-\sqrt[c]{\log x}}). \tag{17}$$

Hierin ist die (nicht negative) zweite Summe

$$\leqq \sum_{n=x+1}^{(1+\delta)x} c_n \log \frac{(1+\delta)x}{x} = \log(1+\delta) \sum_{n=x+1}^{(1+\delta)x} c_n \leqq \delta \sum_{n=x+1}^{(1+\delta)x} c_n. \tag{18}$$

Nun ist bekanntlich†

$$\sum_{n=1}^{x} 2^{\nu(n)} = Ax \log x + Bx + O(\sqrt{x} \log x),$$

wo A und B zwei Constanten sind; also ist a fortiori

$$\begin{aligned}\sum_{n=x+1}^{(1+\delta)x} c_n &= \sum_{n=x+1}^{(1+\delta)x} 2^{\nu(n)} \Theta_{(4,3)}(n) \leqq \sum_{n=x+1}^{(1+\delta)x} 2^{\nu(n)} \\ &= A(1+\delta)x(\log x + \log(1+\delta)) + B(1+\delta)x - Ax\log x - Bx + O(\sqrt{x}\log x) \\ &= A\delta x \log x + B\delta x + A(1+\delta)x\log(1+\delta) + O(\sqrt{x}\log x) \\ &= A\delta x \log x + O(\delta x) + O(\sqrt{x}\log x) = O(x \log x e^{-\sqrt[2c]{\log x}});\end{aligned}$$

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* Ist die obere Summationsgrenze nicht ganz, so hat n alle bis zu ihr gelegenen ganzen Zahlen zu durchlaufen.

† Vergl. Mertens, "Über einige asymptotischen Gesetze der Zahlentheorie," Journal für die reine und angewandte Mathematik, Bd. 77, 1874, S. 294.

daher ist nach (18) die zweite Summe in (17)

$$= O\left(e^{-\sqrt[2c]{\log x}}x \log x e^{-\sqrt[2c]{\log x}}\right) = O\left(x \log x e^{-2\sqrt[2c]{\log x}}\right).$$

Dadurch verwandelt sich (17) in

$$\log(1+\delta)\sum_{n=1}^{x} c_n = \frac{2}{\pi}\,\delta x + O\left(x \log x e^{-2\sqrt[2c]{\log x}}\right);$$

daraus ergiebt sich genau wie a. a. O.*, dass

$$\sum_{n=1}^{x} c_n = \frac{2}{\pi}\,x + O\left(x e^{-\sqrt[\beta]{\log x}}\right)$$

ist, wo β eine positive Constante bezeichnet.

Hierin liegt der von Herrn Lehmer vermutete Satz

$$\lim_{x=\infty}\frac{\sum\limits_{n=1}^{x} c_n}{x} = \lim_{x=\infty}\frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta_{(4,\,3)}(n)}{x} = \frac{2}{\pi}$$

als Spezialfall enthalten.

§7.

Ganz analog lässt sich der Nachweis führen, dass die Grenzwerte

$$\lim_{x=\infty}\frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta_{(3,\,2)}(n)}{x}$$

und

$$\lim_{x=\infty}\frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta_{(6,\,5)}(n)}{x}$$

existieren und gleich $\frac{2\sqrt{3}}{\pi}$ bezw. $\frac{2\sqrt{3}}{3\pi}$ sind. Es gelten nämlich alle Entwickelungen der §§2–6, wenn dabei folgende Aenderungen in der Bezeichnungsweise und in Einzelheiten angebracht werden.

Es werde

$$c_n = 2^{\nu(n)}\Theta_{(3,\,2)}(n)\ [\text{bezw. } c_n = 2^{\nu(n)}\Theta_{(6,\,5)}(n)]$$

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* L. c., S. 530–531.

gesetzt; wenn $g(s)$ die für $\sigma > 1$ durch die Reihe

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

definierte Function bezeichnet, so ist für $\sigma > 1$

$$g(s) = \prod_q \frac{1+\frac{1}{q^s}}{1-\frac{1}{q^s}},$$

wo q alle Primzahlen $3m+2$ [bezw. $6m+5$] durchläuft. Wenn r alle Primzahlen $6m+1$ durchläuft, so ergiebt sich wie in §2, dass

$$g(s) = \left(1-\frac{1}{3^s}\right)\zeta(s)\prod_q\left(1+\frac{1}{q^s}\right)\prod_r\left(1-\frac{1}{r^s}\right) \tag{19}$$

[bezw.

$$g(s) = \left[\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\zeta(s)\prod_q\left(1+\frac{1}{q^s}\right)\prod_r\left(1-\frac{1}{r^s}\right)\right] \tag{20}$$

ist.

Es sei nun $\chi(n)$ so definiert, dass

$$\chi(3m) = 0, \quad \chi(3m+1) = 1, \quad \chi(3m+2) = -1$$

[bezw.

$$\chi(6m) = 0, \quad \chi(6m+1) = 1, \quad \chi(6m+2) = 0,$$
$$\chi(6m+3) = 0, \quad \chi(6m+4) = 0, \quad \chi(6m+5) = -1]$$

ist; dann ist nach (19) [bezw. (20)] für $\sigma > 1$

$$g(s) = \left(1-\frac{1}{3^s}\right)\zeta(s)\prod_p\left(1-\frac{\chi(p)}{p^s}\right) = \left(1-\frac{1}{3^s}\right)\zeta(s)\frac{1}{\sum\limits_{n=1}^{\infty}\frac{\chi(n)}{n^s}}$$

[bezw.

$$g(s) = \left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\zeta(s)\prod_p\left(1-\frac{\chi(p)}{p^s}\right)$$
$$= \left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\zeta(s)\frac{1}{\sum\limits_{n=1}^{\infty}\frac{\chi(n)}{n^s}}\Bigg].$$

Also ist $g(s)$ über die Gerade $\sigma = 1$ fortsetzbar und hat in $s = 1$ einen Pol erster Ordnung mit dem Residuum

$$(1-\tfrac{1}{3})\frac{1}{1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\dots} = \frac{2}{3}\,\frac{1}{\frac{\pi}{9}\sqrt{3}} = \frac{2\sqrt{3}}{\pi}$$

[bezw.

$$(1-\tfrac{1}{2})(1-\tfrac{1}{3})\frac{1}{1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\dots} = \frac{1}{3}\,\frac{1}{\frac{\pi}{6}\sqrt{3}} = \frac{2\sqrt{3}}{3\pi}\Big].$$

Der in §3 citierte Hilfssatz II. gilt auch für die hier vorliegenden beiden Reihen

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

wie a. a. O.* bewiesen ist; daraus folgt für die jetzigen Bedeutungen von $g(s)$ gleichfalls die Richtigkeit des oben auf S. 217–218 ausgesprochenen Satzes und damit wörtlich wie oben in §§4–6 die Existenz des Grenzwertes

$$\lim_{x=\infty}\frac{\sum_{n=1}^{x} c_n}{x},$$

welcher gleich dem Residuum von $g(s)$ im Pole $s = 1$, also gleich $\frac{2\sqrt{3}}{\pi}$ $\left[\text{bezw. } \frac{2\sqrt{3}}{3\pi}\right]$ ist.

BERLIN, den 28ten October 1903.

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* L. c., S. 521.

On Hypercomplex Number Systems in Seven Units.

By H. E. Hawkes.

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§1.

In a paper which recently appeared in the Mathematische Annalen I have given a method for enumerating all distinct classes of non-quaternion number systems in n units, at least two of which are idempotent, from systems in $n-1$ units. This paper supplements those of Starkweather* so that the enumeration problem for non-quaternion systems has now reached a complete solution. I hope in a future paper to complete the enumeration problem by giving a general enumeration of quaternion systems and to apply the complete result to various related subjects. In order to place in available form a source from which examples and illustrations may be drawn, and to place on record the actual enumeration of systems of as high an order as seems at present desirable, I have in the present paper given the enumeration of distinct types of irreducible, non-reciprocal, non-quaternion systems with moduli in seven units at least two of which are idempotent.† In §2 a general device is obtained which removes the necessity for considerable ineffective labor in making the enumeration which my outline in the Annalen required.

§2.

Method of Obtaining Tables of Combination.

An outline of the method to be followed in the enumeration is indicated by the following theorems.‡

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* American Journal of Mathematics, Vols. 21 and 23.

† Systems in less than five units are enumerated by Scheffers, Mathematische Annalen, Vol. 39. Systems in six units, in one idempotent unit, are enumerated by Starkweather, American Journal of Mathematics, Vol. 23. Those in more than one idempotent unit I have given, loc. cit.

‡ Proofs of these theorems may be found in my papers in Transactions of the American Mathematical Society, Vol. 3, and Mathematische Annalen, loc. cit.

1. *Every unit in a non-quaternion system falls into one of the four following groups with respect to each of the idempotent units e_k of the system:*

Group I_k contains those units e_i such that

$$e_i e_k = e_k e_i = e_i$$

Group II_k contains those units e_i such that

$$e_i e_k = 0, \quad e_k e_i = e_i$$

Group III_k contains those units e_i such that

$$e_i e_k = e_i, \quad e_k e_i = 0$$

Group IV_k contains those units e_i such that

$$e_i e_k = e_k e_i = 0$$

2. *If two non-quaternion systems do not contain the same number of idempotent units, they are inequivalent.*

Thus we may enumerate all systems with a given number of idempotent units without including those enumerated with a different number of such units.

3. *If S and S' are equivalent there is a one to one correspondance between the idempotent units e_{r+i} and e'_{r+i} $(i = 1, 2, \ldots, n - r)$, such that the number of units in the groups I_k, II_k, III_k, IV_k, and I'_k, II'_k, III'_k, IV'_k are respectively the same where e_k and e'_k are corresponding units.*

This theorem shows that the first step toward enumeration is the formation of a table which gives, for a given value of r the different combinations of non-idempotent units into groups. We erase from this table all combinations that would lead to reducible systems and one of each pair of combinations that would afford reciprocal systems in accordance with the following principles.

4. *The necessary and sufficient condition that a system is reducible is that its modulus falls into parts each of which is the modulus of a certain subsystem.*

5. *Those combinations that are identical except that the number of units in groups II and III with respect to every idempotent unit are mutually interchanged, lead to reciprocal systems.*

Thus if in a given system there are in groups II_k and III_k, λ and λ' units respectively there will be in groups II_k and III_k of the reciprocal system λ' and λ units respectively. This is equivalent to the statement that the multiplication table of two reciprocal systems differ only in the fact that the rows and columns of one are interchanged to obtain the other.

As a guide to the construction of the tables of combination for systems in seven units, I will give here the tables for $n = 3, 4, 5, 6$. It should be noted that the order of units in these tables is entirely unimportant. In fact, it often appears that the order of the units which is used in the final multiplication table of the system must be different from the one that appears in the tables of combination in order to put the multiplication table into Scheffer's normal form* for non-quaternion systems. Thus, for example: for $n = 5$, $r = 3$ below, in the second system it is by no means essential that e_1 is in group I with respect to e_4, e_2 in I with respect to e_5, etc., but merely in the system for which $r = 3$ there must be one unit in I and one in II with respect to one of the idempotent units, and one unit in I and one in III with respect to the other idempotent unit.

In the following tables the idempotent units are represented at the top and the non-idempotent units at the left-hand side. When there is no ambiguity, the unit e_k is represented by the subscript k only. The group of e_i with respect to e_k is at the intersection of the k^{th} column and i^{th} row. The space where IV would appear is left blank. As usual, r represents the number of non-idempotent units, and k represents the total number of units in all groups I.

$n = 3$

$r = 1, \; k = 0,$

	2	3
1	II	III

* See my paper, Annalen, loc. cit.

$n = 4$

$r = 2, \; k = 1.$

In this case there is one even unit and the combination given below is the only one that is admissible. The combination

	3	4
1	I	
2	III	II

gives systems reciprocal to the one retained while

	3	4
1		I
2	II	III

	3	4
1		I
2	III	II

are equivalent to the combinations given below and above respectively after an interchange of the units 3 and 4.

The table for $k = 0$ is of obvious construction. We have then

	$k = 1$		$k = 0$			
	3	4	3	4	3	4
1	I		II	III	II	III
2	II	III	II	III	III	II

$n = 5$

$r = 2$

In this case we can have no even unit as the system would then be reducible. For one skew unit could not connect three idempotent units, and the modulus would fall apart (see 4 p. 224). Thus we have only

$k = 0$

	3	4	5	3	4	5
1	II	III		II	III	
2		II	III		III	II

$r = 3$

	$k = 2$				$k = 1$				$k = 0$			
	4	5	4	5	4	5	4	5	4	5	4	5
1	I		I		I		I		II	III	II	III
2	I			I	II	III	II	III	II	III	II	III
3	II	III	II	III	II	III	III	II	II	III	III	II

$n = 6$

$r = 3$

$k = 1$

	4	5	6	4	5	6	4	5	6	4	5	6
1	I			I				I			I	
2	II	III		II	III		II	III		II	III	
3		II	III		III	II		II	III		III	II

$k = 0$

	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6
1	II	III		II	III			II	III	II		III	III		II
2	II	III		II	III		II	III		II	III		II	III	
3		II	III		III	II		III	II		II	III		II	III

$r=4$

$k=3$ $\qquad$ $k=2$ $\qquad$ $k=1$ $\qquad$ $k=0$

	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6
1	I		I		I		I		I		I		I		I		II	III	II	III	II	III
2	I		I		I		I			I		I	II	III	II	III	II	III	II	III	II	III
3	I			I	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III	III	II
4	II	III	II	III	II	III	III	II	II	III	III	II	II	III	III	II	II	III	III	II	III	II

In the combinations given above, it is to be noted that in setting up combinations for the case where

(A) $$n=n_1;\quad r=r_1;\quad k=k_1 \neq 0,$$

we may make use of the combinations for the case where

(B) $$n=n_1-k_1;\quad r=r_1-k_1;\quad k=0.$$

For example, the combinations for $n=6$, $r=4$, $k=1$ follow immediately from those for $n=5$, $r=3$, $k=0$. In fact, all possible distinct-combinations of the r_1-k_1 skew units which are called for in case (A) are already obtained in case (B). Thus, to get all possible distinct combinations for case (A) we only need to arrange the k_1 even units so as to give the various distinct combinations. This is illustrated in the tables given above in the construction of the case $n=6$, $r=4$, $k=2$ from $n=4$, $r=2$, $k=0$. This principle is also applicable when $k=0$. For instance, if we wish to write the combinations for

$$n=n_1;\quad r=r_1;\quad k=0,$$

we may make use of the table for the case

$$n=n_1-1;\quad r=r_1-1;\quad k=0,$$

which gives all possible combinations of r_1-1 skew units, and the proper arrangement of the remaining skew unit with respect to these, is all that remains. This is illustrated in the derivation of the combination for $n=6$, $r=4$, $k=0$ from those for $n=5$, $r=3$, $k=0$. By this device it is possible

to write the tables of combination very rapidly by inspection without including any that are superfluous.

§3.

Tables of Combination for $n = 7$.

Since two skew units can connect at most three idempotent units, all systems for which $r < 3$ are reducible. Thus we first consider

$r = 3$.

There are in this case four idempotent units, a number which is not reached in irreducible systems in less than seven units. Thus for this case the tables of combination must be constructed without the assistance of tables already derived. If one of our three non-idempotent units is even, it leaves only two skew units to connect four idempotent units, which is impossible. Thus all the non-idempotent units must be skew, and we obtain

$k = 0$

	1_3				2_3				3_3				4_3				5_3			
	4	5	6	7	4	5	6	7	4	5	6	7	4	5	6	7	4	5	6	7
1	II	III			II	III			II	III			II	III			II	III		
2	II		III		II		III		II		III		II		III		III		II	
3	II			III	III			II		III		II		II		III		II		III

$r = 4$.

Since two skew units are sufficient to connect three idempotent, we may have in this case at most two even units. For the case $k = 2$ we make use of the table for $n = 5$, $r = 2$, $k = 0$.

$k=2$

	1_4			2_4			3_4			4_4			5_4			6_4			7_4			8_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1	I			I				I			I		I			I			I			I		
2	I			I				I			I			I			I				I			I
3	II	III		II	III		II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		II	III		III	II		II	III		III	II		II	III		III	II

$k=1$

Here we make use of the case $n=6$, $r=3$, $k=0$.

	9_4			10_4			11_4			12_4			13_4			14_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1	I			I			I			I			I				I	
2	II	III		II	III			II	III	II		III	III		II	II	III	
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		III	II		II	III		II	III		II	III

	15_4			16_4			17_4			18_4			19_4			20_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1		I			I			I				I			I			I
2	II	III			II	III	II		III	II	III		II	III			II	III
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		III	II		III	II		II	III		II	III		III	II		III	II

$k = 0$

In this case we also use $n = 6$, $r = 3$, $k = 0$.

	21_4			22_4			23_4			24_4			25_4			26_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1	II	III		II	III		II	III		II	III		II		III	II		III
2	II	III		II	III			II	III	II		III	II	III			II	III
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		III	II		II	III		III	II		III	II

	27_4			28_4			29_4			30_4			31_4			32_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1		II	III		II	III	III	II		III	II		III		II		III	II
2	II	III			II	III	II	III			II	III	II	III		II	III	
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		III	II		III	II		II	III		III	II

$r = 5$.

By use of combinations for $n = 3$, $r = 1$, $k = 0$ and $n = 4$, $r = 2$, $k = 0$ we obtain combinations for $k = 4$ and $k = 3$.

$k = 4$ $k = 3$

	1_5		2_5		3_5		4_5		5_5		6_5		7_5	
	6	7	6	7	6	7	6	7	6	7	6	7	6	7
1	I		I		I		I		I		I		I	
2	I		I		I		I		I		I		I	
3	I		I			I	I		I			I		I
4	I			I		I	II	III	II	III	II	III	II	III
5	II	III	II	III	II	III	II	III	III	II	II	III	III	II

By use of combinations for $n = 5$, $r = 3$, $k = 0$ and $n = 6$, $r = 4$, $k = 0$ we obtain the combinations for $k = 2$, $k = 1$ and $k = 0$.

$k = 2$ $k = 1$ $k = 0$

	8_5		9_5		10_5		11_5		12_5		13_5		14_5		15_5		16_5		17_5	
	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7
1	I		I		I		I		I		I		I		II	III	II	III	II	III
2	I		I			I		I	II	III	II	III	II	III	II	III	II	III	II	III
3	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III
4	II	III	II	III	II	III	II	III	II	III	II	III	III	II	II	III	II	III	III	II
5	II	III	III	II	II	III	III	II	II	III	III	II	III	II	II	III	III	II	III	II

§4.

Method of Obtaining Multiplication Tables.

In any non-quaternion system the units can be chosen so that the multiplication table is in Scheffers' normal form. This form is characterized as follows. Let $e_1, e_2, \ldots, e_r$ be the non-idempotent units.

Then
$$e_ie_j = \sum_{k=1}^{l-1} \gamma_{ijk} e_k \qquad i, j \leqq r$$

where l is the lesser of i and j. Also

$$e_ie_{r+s} = e_{r+s}e_i = 0 \qquad i \leqq r;\ 0 < s \leqq n-r$$

except when e_i is in groups I, II or III with respect to e_{r+s} when the value of the product is determined by the group in which e_i is found with respect to e_{r+s}. Also,

$$e_{r+s}\, e_{r+t} = 0, \qquad s \neq t,\ 0 < s,\ t \leqq n-r.$$
$$e_{r+s}\, e_{r+s} = e_{r+s}, \qquad 0 < s \leqq n-r.$$

The units of any system in which the independent idempotent numbers are taken as units, are also subject to multiplicative properties expressed by the following table, which gives the group to which the product (when non-vanishing) of two units of given groups must belong:

(A)

	I	II	III	IV
I	I	II	0	0
II	0	0	I	II
III	III	IV	0	0
IV	0	0	III	IV

Thus, for instance, if e_i and e_j are in groups II and III respectively with respect to a given idempotent unit, their product e_ie_j will be in group I with respect to the same unit, provided it does not vanish.

Thus the units of any non-quaternion system must fall in Scheffers' normal form and obey table (A) simultaneously. Consequently the first step in deriving the multiplication tables from the tables of combination is to apply the associative law to bring the units into Scheffers' normal form. The units are assumed to obey table (A) at the outset. This process is always possible, and very simple to carry out. For further reduction of parameters we make use of the following principles:

Definition. A system is said to be deleted by a given unit, when that unit is erased from every position which it occupies in the multiplication table of the system.

Definition. A number is nilfactorial with respect to a number β if

$$\alpha\beta = \beta\alpha = 0.$$

6. *If a non-quaternion system is deleted by a unit that is nilfactorial with respect to every non-idempotent unit, the deleted system is associative.*

7. *If a system is deleted by one or more units so that there remains only certain idempotent units and one or more unbroken groups with respect to those units, the deleted system is associative.*

8. *If two systems are deleted by the same method (i. e. both under 6) and the deleted systems are inequivalent, the original systems are inequivalent.*

Principles 7 and 8 show that when in a table of combination, for instance, three units of group I with respect say to e_7 $(n = 7)$ occur, that the various distinct systems in four units, three of which are in group I with respect to the remaining unit, constitute sub-systems of distinct systems in seven units.

The types of distinct systems in less than six units are taken from Scheffers' paper already quoted.

§5.

Enumeration of Inequivalent Systems in Seven Units.

The tables of combinations given in §3 determine the portions of the corresponding multiplication tables which involve the idempotent units in products either with each other or with the non-idempotent units. Thus 1_3 leads to the table following where table (A) has been applied to determine the products

of non-idempotent units, and the system is assumed to be in Scheffers' normal form.

	1	2	3	4	5	6	7
1	0	0	0	0	1	0	0
2	0	0	0	0	0	2	0
3	0	0	0	0	0	0	3
4	1	2	3	4	0	0	0
5	0	0	0	0	5	0	0
6	0	0	0	0	0	6	0
7	0	0	0	0	0	0	7

In the following enumeration the portion of the multiplication table involving the idempotent units is not displayed in matricular form as above, but is to be supplied by the reader from the tables of combination. The non-vanishing products of non-idempotent units are given, which, together with the corresponding table of combination, determine the systems completely. Thus "all vanish" means that all products of non-idempotent units among themselves vanish, while "$e_2e_2 = e_1$" means that all products of non-idempotent units among themselves with this exception vanish.

Thus:

$r = 0$

$k = 0$

1_3. All vanish.

2_3. All vanish.

3_3. All vanish.

4_3. All vanish.

5_3. All vanish.

$r = 4$

$k = 2$. 1_4. Since $\mathrm{I} \cdot \mathrm{II} = \mathrm{II}$ by table (A), and since no product of two non-idempotent units can contain either one of those units, we have $\mathrm{I} \cdot \mathrm{II} = 0$. Similarly $\mathrm{II} \cdot \mathrm{III} = \mathrm{I}$, but as there is no unit in I with respect to e_6, we have

II · III = 0. Thus, by 8 and systems III_1 and III_2 of Scheffers' enumeration, we have

$1_4 \cdot 1$. $e_2e_2 = e_1$.
$1_4 \cdot 2$. All vanish.
$2_4 \cdot 1$. $e_2e_2 = e_1$.
$2_4 \cdot 2$. All vanish.
$3_4 \cdot 1$. $e_2e_2 = e_1$.
$3_4 \cdot 2$. All vanish.
4_4. All vanish.
5_4. All vanish.
6_4. All vanish.
7_4. All vanish.
8_4. All vanish.

$k = 1$. 9_4. In this case, if we take the units in the order given in the table of combination, we should have $e_1e_3 = e_2$, which is not according to Scheffers' normal form. But on interchange of the units e_1 and e_2 we have an equivalent system,

$9_4 \cdot 1$. $e_2e_3 = e_1$.
$9_4 \cdot 2$. All vanish.
10_4. Interchange e_1 and e_2 and we obtain
$10_4 \cdot 1$. $e_2e_3 = e_1$.
$10_4 \cdot 2$. All vanish.
11_4. All vanish.
$12_4 \cdot 1$. $e_3e_4 = e_2$.
$12_4 \cdot 2$: All vanish.
13_4. All vanish.
14_4. Interchange e_1 and e_2 and we obtain
$14_4 \cdot 1$. $e_3e_2 = e_1$.
$14_4 \cdot 2$. All vanish.
15_4. Interchange e_1 and e_2 and we obtain
$15_4 \cdot 1$. $e_3e_2 = e_1$.
$15_4.2$. All vanish.
$16_4 \cdot 1$. $e_2e_4 = e_1$.
$16_4 \cdot 2$. All vanish.
$17_4 \cdot 1$. $e_3e_4 = e_1$.

$17_4 \cdot 2$. All vanish.
18_4. All vanish.
19_4. All vanish.
$20_4 \cdot 1$. $e_4e_2 = e_1$.
$20_4 \cdot 2$. All vanish.
$k = 0$, 21_4. All vanish.
22_4. All vanish.
23_4. All vanish.
$24_4 \cdot 1$. $e_3 \cdot e_4 = e_2$.
$24_4 \cdot 2$. All vanish.
25_4. Interchange e_1 and e_2 and we obtain
$25_4 \cdot 1$. $e_2e_4 = e_1$.
$25_4 \cdot 2$. All vanish.
$26_4 \cdot 1$. $e_3e_2 = e_1$.
$26_4 \cdot 2$. All vanish.
27_4. All vanish.
28_4. All vanish.
29_4. All vanish.
30_4. All vanish.
31_4. All vanish.
32_4. All vanish.

$r = 5$.

$k = 4$. 1_5. We get as many inequivalent systems as there are inequivalent systems in five units one of which is idempotent.

Thus, $1_5 \cdot 1$. $e_2e_4 = e_4e_2 = e_3^2 = e_1$; $e_3e_4 = e_4e_3 = e_2$; $e_4^2 = e_3$.
$1_5 \cdot 2$. $e_2e_3 = e_3e_2 = e_3e_4 = - e_4e_3 = e_4^2 = e_1$; $e_3^2 = e_2$.
$1_5 \cdot 3$. $e_2e_3 = e_3e_2 = e_4^2 = e_1$; $e_3^2 = e_2$.
$1_5 \cdot 4$. $e_2e_3 = e_3e_2 = e_3e_4 = - e_4e_3$; $e_3^2 = e_2$.
$1_5 \cdot 5$. $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$.
$1_5 \cdot 6$. $e_3e_4 = \frac{1}{\lambda} e_4e_3 = e_1$; $e_4^2 = e_2$.
$1_5 \cdot 7$. $e_3^2 = e_1$; $e_4^2 = e_2$.
$1_5 \cdot 8$. $e_3e_4 = e_1$; $e_4e_3 = e_2$.
$1_5 . 9$. $e_3e_4 = e_1 + e_2$; $e_4e_3 = - e_1 + e_2$; $e_4^2 = e_1$.

$1_5{\cdot}10.$ $e_3^2 = e_1$; $e_3e_4 = -e_4e_3 = e_2$; $e_4^2 = e_2 + \lambda e_1$.
$1_5{\cdot}11.$ $e_2^2 = e_3^2 = e_4^2 = e_1$.
$1_5.12.$ $e_3^2 = e_4^2 = e_1$.
$1_5{\cdot}13.$ $e_4^2 = e_1$.
$1_5{\cdot}14.$ $e_2^2 = e_3^2 = e_4^2 = \frac{1}{\lambda} e_3e_4 = -\frac{1}{\lambda} e_4e_3 = e_1$.
$1_5{\cdot}15.$ $e_2^2 = e_3e_4 = -e_4e_3 = e_4^2 = e_1$.
$1_5{\cdot}16.$ $e_2^2 = e_3e_4 = -e_4e_3 = e_1$,
$1_5{\cdot}17.$ $e_3^2 = e_4^2 = \frac{1}{\lambda} e_3e_4 = -\frac{1}{\lambda} e_4e_3 = e_1$.
$1_5{\cdot}18.$ $e_3e_4 = -e_4e_3 = e_4^2 = e_1$.
$1_5{\cdot}19.$ $e_2e_3 = e_3e_2 = e_4e_3 = -e_4e_3 = e_4^2 = e_1$.
$1_5{\cdot}20.$ $e_2e_3 = e_3e_2 = e_4e_3 = -e_4e_3 = e_1$.
$1_5{\cdot}21.$ $e_3e_4 = -e_4e_3 = e_1$.
$1_5{\cdot}22.$ All vanish.

2_5. We make similar use of Scheffers' list of systems in four units and obtain,
$2_5{\cdot}1.$ $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$.
$2_5{\cdot}2.$ $e_2^2 = e_2e_3 = -e_3e_2 = \frac{1}{\lambda} e_3^2 = e_1$.
$2_5{\cdot}3.$ $e_2^2 = e_3^2 = e_1$.
$2_5{\cdot}4.$ $e_3^2 = e_1$.
$2_5{\cdot}5.$ $e_2e_3 = -e_3e_2 = e_1$.
$2_5{\cdot}6.$ All vanish.

$3_5{\cdot}1.$ $e_2^2 = e_1$; $e_4^2 = e_3$.
$3_5{\cdot}2.$ $e_2^2 = e_1$.
$3_5{\cdot}3.$ All vanish.

$k = 3$ $4_5{\cdot}1.$ We get by use of the associative law on the products $e_2e_3e_5$ and $e_3^2e_5$, after interchanging e_3 and e_4.
$4_5{\cdot}1.$ $e_2e_4 = e_4e_2 = e_1$; $e_4^2 = e_2$; $e_4e_5 = e_3$.
$4_5{\cdot}2.$ $e_2e_4 = e_4e_2 = e_1$; $e_4^2 = e_2$.

$4_5{\cdot}3$–14. Interchange e_4 and e_2 and obtain

$4_5{\cdot}3$. $e_4^2 = e_4e_3 = -e_3e_4 = \frac{1}{\lambda}e_3^2 = e_1$; $e_4e_5 = e_2$; $e_3e_5 = \lambda_1e_2$.

$4_5{\cdot}4$. $e_4^2 = e_4e_3 = -e_3e_4 = \frac{1}{\lambda}e_3^2 = e_1$; $e_3e_5 = e_2$.

$4_5{\cdot}5$. $e_4^2 = e_4e_3 = -e_3e_4 = \frac{1}{\lambda}e_3^2 = e_1$.

$4_5{\cdot}6$. $e_4^2 = e_3^2 = e_1$; $e_4e_5 = \frac{1}{\lambda}e_3e_5 = e_2$.

$4_5{\cdot}7$. $e_4^2 = e_3^2 = e_1$; $e_4e_5 = e_2$.

$4_5{\cdot}8$. $e_4^2 = e_3^2 = e_1$.

$4_5{\cdot}9$. $e_3^2 = e_1$; $e_4e_5 = e_2$.

$4_5{\cdot}10$. $e_3^2 = e_1$.

$4_5{\cdot}11$. $e_4e_3 = -e_3e_4 = e_1$; $e_3e_5 = e_2$.

$4_5{\cdot}12$. $e_4e_3 = -e_3e_4 = e_1$.

$4_5{\cdot}13$. $e_4e_5 = e_2$.

$4_5{\cdot}14$. All vanish.

5_5. No interchange is required.

$5_5{\cdot}1$. $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$; $e_4e_5 = e_1$.

$5_5{\cdot}2$. $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$.

$5_5{\cdot}3$. $e_2e_3 = -e_3e_2 = e_2^2 = \frac{1}{\lambda}e_3^2 = e_1$; $e_4e_5 = e_1$.

$5_5{\cdot}4$. $e_2e_3 = -e_3e_2 = e_2^2 = \frac{1}{\lambda}e_3^2 = e_1$.

$5_5{\cdot}5$. $e_2^2 = e_3^2 = e_4e_5 = e_1$.

$5_5{\cdot}6$. $e_2^2 = e_3^2 = e_1$.

$5_5{\cdot}7$. $e_3^2 = e_1$; $e_4e_5 = e_2$.

$5_5{\cdot}8$. $e_3^2 = e_1$; $e_4e_5 = e_1$.

$5_5{\cdot}9$. $e_3^2 = e_1$.

$5_5{\cdot}10$. $e_2e_3 = -e_3e_2 = e_4e_5 = e_1$.

$5_5{\cdot}11$. $e_2e_3 = -e_3e_2 = e_1$.

$5_5{\cdot}12$. $e_4e_5 = e_1$.

$5_5{\cdot}13$. All vanish.

6_5·1–6. Interchange e_2 and e_4.
6_5·1. $e_4^2 = e_1$; $e_4e_5 = e_2$; $e_5e_3 = e_2$.
6_5·2. $e_4^2 = e_1$; $e_4e_5 = e_2$.
6_5·3. $e_4^2 = e_1$; $e_5e_3 = e_2$.
6_5·4 $e_4^2 = e_1$.
6_5·5. $e_4e_5 = e_2$; $e_5e_3 = e_2$.
6_5·6. $e_4e_5 = e_2$.
6_5·7. All vanish.

7_5·1. $e_2^2 = e_1$; $e_5e_4 = e_3$; $e_4e_5 = e_1$.
7_5·2. $e_2^2 = e_1$; $e_5e_4 = e_3$.
7_5·3. $e_2^2 = e_1$; $e_4e_5 = e_1$.
7_5·4. $e_2^2 = e_1$.
7_5·5. $e_4e_5 = e_1$; $e_5e_4 = e_3$.
7_5·6. $e_4e_5 = e_1$.
7_5·7. All vanish.

$k = 2$ 8_5·1–9. Interchange cyclically $e_1e_4e_2e_5e_3$.
8_5·1. $e_5^2 = e_4$; $e_4e_3 = e_5e_2 = e_1$; $e_5e_3 = e_2$.
8_5·2. $e_5^2 = e_4$; $e_5e_3 = e_2$.
8_5·3. $e_5^2 = e_4$; $e_5e_2 = e_1$.
8_5·4. $e_5^2 = e_4$.
8_5·5. $e_4e_3 = e_1$; $e_5e_3 = e_2$.
8_5·6. $e_4e_3 = e_1$; $e_5e_2 = e_1$.
8_5·7. $e_4e_3 = e_1$.
8_5·8. All vanish.

9_5.1–8. Interchange e_2 and e_3.
9_5·1. $e_3^2 = e_1$; $e_4e_5 = e_1$.
9_5·2. $e_3^2 = e_1$.
9_5·3. $e_3^2 = e_1$; $e_3e_4 = e_2$; $e_4e_5 = e_1$.
9_5·4. $e_3^2 = e_1$; $e_3e_4 = e_2$.
9_5·5. $e_3e_4 = e_2$; $e_4e_5 = e_1$.
9_5·6. $e_3e_4 = e_2$.

$9_5 \cdot 7$. $e_3e_5 = e_1$; $e_4e_5 = e_3$.
$9_5 \cdot 8$. $e_3e_5 = e_1$.
$9_5 \cdot 9$. All vanish.

$10_5 \cdot 1$–6. Interchange e_1 with e_3, and e_2 with e_4.
$10_5 \cdot 1$. $e_3e_5 = e_2$; $e_5e_4 = e_1$.
$10_5 \cdot 2$. $e_3e_5 = e_2$; $e_5e_4 = e_2$.
$10_5 \cdot 3$. $e_3e_5 = e_2$.
$10_5 \cdot 4$. $e_3e_2 = e_1$; $e_5e_4 = e_1$.
$10_5 \cdot 5$. All vanish.

$11_5 \cdot 1$–3. Interchange e_1 and e_3.
$11_5 \cdot 1$. $e_3e_4 = e_1$; $e_4e_2 = e_1$.
$11_5 \cdot 2$. $e_3e_4 = e_1$; $e_5e_4 = e_2$.
$11_5 \cdot 3$. $e_3e_4 = e_1$.
$11_5 \cdot 4$. $e_4e_5 = e_1$; $e_5e_3 = e_2$.
$11_5 \cdot 5$. $e_4e_5 = e_1$.
$11_5 \cdot 6$. $e_4e_5 = e_1$; $e_5e_4 = e_2$.
$11_5 \cdot 7$. All vanish.

$k = 1$ $12_5 \cdot 1$–3. Interchange cyclically $e_1e_4e_3e_2$.
$12_5 \cdot 1$. $e_4e_2 = e_1$; $e_4e_5 = e_3$.
$12_5 \cdot 2$. $e_4e_5 = e_3$.
$12_5 \cdot 3$. All vanish.

$13_5 \cdot 1$–3. Interchange e_1 and e_2.
$13_5 \cdot 1$. $e_2e_3 = e_1$.
$13_5 \cdot 2$. $e_2e_4 = e_1$; $e_3e_5 = e_2$.
$13_5 \cdot 3$. $e_4e_5 = e_2$.
$13_5 \cdot 4$. All vanish.

$14_5 \cdot 1$. Interchange e_1 and e_2.
$14_5 \cdot 1$. $e_1e_3 = e_2$.
$14_5 \cdot 2$. Interchange cyclically $e_1e_2e_4$.
$14_5 \cdot 2$. $e_4e_5 = e_1$; $e_5e_2 = e_1$.

	$14_5{\cdot}3$.	$e_2e_5 = e_1$; $e_3e_4 = e_1$.
	$14_5{\cdot}4$.	$e_2e_5 = e_1$.
	$14_5{\cdot}5$.	All vanish.
$k = 0$	15_5.	All vanish.
	16_5.	All vanish.
	17_5.	All vanish.

YALE UNIVERSITY, *July* 29, 1901.

Memoir on Abelian Transformations.

By Leonard Eugene Dickson.

Introduction.

One of the most important concepts relating to a group is the distribution of all of its operators into complete sets of conjugates within the group. It is presupposed, for example, in the recent theory of group-characters. For the general linear homogeneous group in an arbitrary field, this problem is readily treated* by means of the theory of canonical forms, as established by Jordan† for a field of prime order and by another method by the writer,‡ first for any finite field and later for an arbitrary field. By a supplementary discussion, the same method suffices for the group of linear homogeneous (or fractional) transformations of determinant unity, for the case of a finite field.|| For binary transformations, the case of an arbitrary field is treated in §1.

When we pass to a special class of groups, as the linear Abelian group, the problem is incomparably more difficult. The Abelian group affects an even number $2m$ of variables. The case $m = 1$ is treated in §1. The case $m = 2$ has been treated for a Galois field by the writer.§ The present paper aims to present a systematic general method of treatment for an arbitrary field, and to present in complete form the desired numerical results for $m = 3$ and the $GF[p^n]$. The paper presents a generalization to an arbitrary field of the canonical forms for $m = 2$, but does not duplicate the numerical work given in

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* Simple generalization of the papers by Dickson and Putnam on the ternary and quaternary groups, American Journal of Mathematics, Vol. XXIII, p. 37.

† Traité des Substitutions, pp. 114–126.

‡ American Journal, Vol. XXII, pp. 121–7; Vol. XXIV, pp. 101–8.

|| Dickson, American Journal, Vol. XXII, pp. 231–252; Putnam, ibid., Vol. XXIV.

§ Trans. Amer. Math. Soc., Vol. 2 (1901), pp. 103–138. Referred to as Tr. I had occasion to recheck all of the results of that paper.

the Transactions paper. The treatment in §§18–19 is much simpler than in the former paper. Moreover, §20, case (i), gives a necessary correction.* A few extra pages of this memoir therefore properly deal with this case $m=2$. Moreover, a large part of the work for $m=3$ consists of proof by induction from $m=1$ and $m=2$ to $m=3$.

In order to allow a more rapid and connected determination of the canonical types, I have reserved the determination of the number of non-conjugate forms of each type and the number of commutative transformations to a series of consecutive sections (§§37–42), which may conveniently serve for reference in place of the usual summarizing tables.

In certain cases I have found the completion of the discussion for an arbitrary field to be impracticable. I believe it highly desirable that an investigation should be made as far as expedient for an arbitrary domain, with explicit notice when specialization to a particular field (complex, real, rational, finite, ...) is made to secure needed properties.

As an incidental result, it may be noted that the senary Abelian group contains subgroups isomorphic with the general ternary linear group (§37), general ternary hyperorthogonal group (§40, type $T_{1\lambda}T_{2\lambda}T_{3\lambda}$), and the ternary orthogonal group (§39, type $L_{11}L_{21}L_{3\mu}$).

As to the importance of the linear Abelian group, one may mention the application to hyperelliptic and Abelian functions, and to various geometrical problems such as the 27 lines on a general cubic surface (group for $m=2$ $p^n=3$), the 28 bitangents to a quartic curve (group for $m=3$, $p^n=2$, a complete table for whose operators is given at the end of the paper).

If the modular theory is destined to be carried to a higher stage, the linear Abelian group in a finite field will play the same role as the congruence groups do in relation to the modular group.

I should state that the detailed steps of this paper have been carefully checked, while the considerably condensed character of §§37–41, which give the final results, led me to re-check this part. Very convincing checks are noted in §42.

1. Frequent use will be made of the canonical forms and conjugacies of

* The statement (Tr., middle p. 129) that T replaces ϕ by ϕ'.

binary transformations of determinant unity in a general field F. Moreover, the group G of all such binary transformations is the group of binary special Abelian transformations (§2 for $m = 1$).

Consider the transformation with coefficients in F

$$T:\quad x' = \alpha x + \gamma y, \quad y' = \beta x + \delta y, \qquad (\alpha\delta - \beta\gamma = 1).$$

Its characteristic equation is

$$\Delta(\rho) \equiv \rho^2 - d\rho + 1 = 0, \quad d \equiv \alpha + \delta.$$

Denote its roots by $\varkappa, \varkappa^{-1}$. Suppose first that $\varkappa$ belongs to F. Then according as $\varkappa \neq \varkappa^{-1}$ or $\varkappa = \varkappa^{-1}$, T is conjugate within G with the respective type

$$A:\quad x' = \varkappa x, \quad y' = \varkappa^{-1} y;$$
$$B_\beta:\quad x' = \pm x, \quad y' = \pm y + \beta x.$$

Now $B_\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} B_\delta$ if, and only if,

$$b\beta = b\delta = 0, \quad d\beta = a\delta, \quad ad - bc = 1.$$

If $\delta \neq 0$, B_δ is not conjugate with B_0. If $\delta \neq 0$, and $\beta \neq 0$, then $b = 0$, $ad = 1$, $d\beta = a\delta$, so that B_β and B_δ are conjugate only if β/δ is the square of an element a of F. *Each B_β is conjugate within G with one of the non-conjugate types B_0, B_1, B_{ν_i}, where ν_i runs through a series of not-squares in F such that no two have as their ratio a square in F, while every not-square has with some ν_i a ratio which is a square. If F be the $GF[p^n]$, there are three types B_0, B_1, B_ν or two types B_0, B_1, according as $p > 2$ or $p = 2$.*

Suppose next that $\Delta(\rho)$ is irreducible in F, so that $\varkappa$ does not lie in F. Then $\gamma \neq 0$, and $y' = y + \alpha/\gamma$ transforms T into

$$S_\gamma:\quad x' = \gamma y, \quad y' = -\gamma^{-1} x + dy.$$

Introducing the new variables, conjugate with respect to F,

$$X = x - \gamma\varkappa y, \quad Y = x - \gamma\varkappa^{-1} y,$$

we obtain for S_γ the canonical form (not belonging to G)

$$C:\quad X' = \varkappa X, \quad Y' = \varkappa^{-1} Y.$$

Hence a transformation which transforms S_γ into S_{γ_1} merely multiplies X and Y by constants, and hence, if it belongs to G, must have the form

$$x' = ax - \gamma\gamma_1 by, \quad y' = bx + (\gamma\gamma_1^{-1}a - \gamma db)\, y,$$

where
$$\gamma\gamma_1^{-1}(a^2 - \gamma_1 dab + \gamma_1^2 b^2) = 1.$$

If $\gamma\gamma_1^{-1}$ is a square in F, the condition may be satisfied by taking $b = 0$. If $\gamma\gamma_1^{-1}$ is a not-square, then $b \neq 0$ and

$$k^2 - kd + 1 = \frac{1}{\gamma\gamma_1 b^2}, \quad k \equiv \frac{a}{b\gamma_1}.$$

For the $GF[p^n]$, $p > 2$, this condition can be satisfied. Indeed, $k^2 - kd + 1$, which is not zero in view of the irreducibility of $\Delta(\rho)$, has at least $\frac{1}{2}(p^n + 1)$ values, one of which is therefore a not-square in the field. For the corresponding value of k and a suitably chosen value of b, the condition will be satisfied. For the field of all real numbers, the condition can be satisfied if, and only if, $k^2 - kd + 1$ can be made negative, viz., if $d^2 > 4$.

The S_γ are all conjugate within F if F is the $GF[p^n]$, or if F is the field R of all real numbers and $d^2 > 4$. For the field R and $d^2 < 4$, the S fall into two distinct sets of conjugates, represented by S_1 and S_{-1}. For a general F, S_γ and S_{γ_1} are conjugate if γ/γ_1 is a square in F or if γ/γ_1 is a not-square ν such that $\nu(k^2 - kd + 1)$ is a square for some value of k in the field F, but not conjugate in the remaining case.

Definitions, Notations, Lemmas, §§2–5.

2. Consider two sets of variables ξ_i, x_i $(i = 1, \ldots, 2m)$ and the function

$$\psi_{\xi, x} = \sum_{l=1}^{m} \begin{vmatrix} \xi_{2l-1} & \xi_{2l} \\ x_{2l-1} & x_{2l} \end{vmatrix}. \tag{1}$$

Let the two sets of variables be transformed cogrediently as follows:

$$\xi_i' = \sum_{j=1}^{2m} \alpha_{ij}\xi_j, \quad x_i' = \sum_{j=1}^{2m} \alpha_{ij} x_j, \quad (i = 1, \ldots, 2m). \tag{2}$$

Introducing the abbreviation, for r, s any distinct integers $\overline{\gtrless}\ 2m$,

$$C_{rs} = \sum_{l=1}^{m} \begin{vmatrix} \alpha_{2l-1\,r} & \alpha_{2l-1\,s} \\ \alpha_{2l\,r} & \alpha_{2l\,s} \end{vmatrix}, \tag{3}$$

involving the elements of the r^{th} and s^{th} columns of (α_{ij}), we find that

$$\psi_{\xi', x'} = \sum_{\substack{r, s = 1, \ldots, 2m \\ r < s}} C_{rs} \begin{vmatrix} \xi_r & \xi_s \\ x_r & x_s \end{vmatrix}. \tag{4}$$

The conditions for the identity $\psi_{\xi', x'} = \mu \psi_{\xi, x}$, μ being constant, are

$$C_{rs} = \left\{ \begin{matrix} \mu \text{ (if } s = r + 1 = \text{even)} \\ 0 \text{ (unless } s = r + 1 = \text{even)} \end{matrix} \right\} (r, s = 1, \ldots, 2m;\ r < s). \tag{5}$$

The totality of transformations (α_{ij}), such that each α_{ij} and μ belong to a field F and satisfy the preceding conditions, constitutes the *general Abelian group* $GA(2m, F)$. Those of its transformations which have $\mu = 1$, and hence leave $\psi_{\xi, x}$ absolutely invariant, form the *special Abelian group* $SA(2m, F)$.

The inverse of a general Abelian transformation $A = (\alpha_{ij})$ is

$$A^{-1}: \left\{ \begin{aligned} \xi'_{2i-1} &= \frac{1}{\mu} \sum_{j=1}^{m} (\alpha_{2j\,2i}\, \xi_{2j-1} - \alpha_{2j-1\,2i}\, \xi_{2j}), \\ \xi'_{2i} &= \frac{1}{\mu} \sum_{j=1}^{m} (-\alpha_{2j\,2i-1}\, \xi_{2j-1} + \alpha_{2j-1\,2i-1}\, \xi_{2j}). \end{aligned} \right\} (i = 1, \ldots, m) \tag{6}$$

Indeed, the product AA^{-1} replaces ξ_{2i-1} by

$$\frac{1}{\mu} \sum_{k=1}^{m} (C_{2k-1\,2i}\, \xi_{2k-1} + C_{2k\,2i}\, \xi_{2k}) \equiv \xi_{2i-1},$$

and likewise replaces ξ_{2i} by ξ_{2i}. Useful relations, together equivalent to the set (5), are obtained by determining directly the conditions that (6) shall make $\psi_{\xi', x'} = \frac{1}{\mu} \psi_{\xi, x}$. Evidently they may be obtained from (5) by replacing $\alpha_{2i-1\,2j-1}$ by $\frac{1}{\mu} \alpha_{2j\,2i}$, etc. The resulting relations (cf. Linear Groups, p. 116) are

$$R_{rs} = \left\{ \begin{matrix} \mu \text{ (if } s = r + 1 = \text{even)} \\ 0 \text{ (unless } s = r + 1 = \text{even)} \end{matrix} \right\} (r, s = 1, \ldots, 2m;\ r < s) \tag{7}$$

where the symbol

$$R_{rs} = \sum_{l=1}^{m} \begin{vmatrix} \alpha_{r\,2l-1} & \alpha_{r\,2l} \\ \alpha_{s\,2l-1} & \alpha_{s\,2l} \end{vmatrix} \tag{8}$$

involves the elements of the r^{th} and s^{th} rows of the matrix (α_{ij}).

3. Lemma. *If, for any transformation* $A=(\alpha_{ij})$, $R_{2t-1\,2t}=\mu_t (t=1, \ldots, m)$ *and the remaining* R_{rs} *are all zero, then, for the inverse* A^{-1}, $C_{2t-1\,2t}=1/\mu_t$ $(t=1, \ldots, m)$ *and the remaining* C_{rs} *are all zero.*

It is first shown that the inverse A^{-1} is

$$\left.\begin{aligned} \xi'_{2i-1} &= \sum_{j=1}^{m} \frac{1}{\mu_j} (\alpha_{2j\,2i}\,\xi_{2j-1} - \alpha_{2j-1\,2i}\,\xi_{2j}), \\ \xi'_{2i} &= \sum_{j=1}^{m} \frac{1}{\mu_j} (-\alpha_{2j\,2i-1}\,\xi_{2j-1} + \alpha_{2j-1\,2i-1}\,\xi_{2j}). \end{aligned}\right\} (i=1, \ldots, m) \quad (9)$$

In proof, we note that the product $A^{-1}A$ replaces ξ_k by

$$\sum_{j=1}^{m} \left(\frac{1}{\mu_{2j-1}} R_{k\,2j}\,\xi_{2j-1} + \frac{1}{\mu_{2j}} R_{2j-1\,k}\,\xi_{2j} \right) = \frac{1}{\mu_k} R_{2t-1\,2t}\,\xi_k = \xi_k,$$

upon setting $k=2t-1$ or $k=2t$, according as k is odd or even.

The values of C_{ij} for A^{-1} may now be derived from the values of the R_{ij} for A. For $j \gtreqless k$, we have from (9)

$$C_{2j-1\,2k} = \sum_{i=1}^{m} \frac{1}{\mu_j \mu_k} \begin{vmatrix} \alpha_{2j\,2i} & -\alpha_{2k-1\,2i} \\ -\alpha_{2j\,2i-1} & \alpha_{2k-1\,2i-1} \end{vmatrix} = \frac{-1}{\mu_j \mu_k} \sum_{i=1}^{m} \begin{vmatrix} \alpha_{2j\,2i-1} & \alpha_{2j\,2i} \\ \alpha_{2k-1\,2i-1} & \alpha_{2k-1\,2i} \end{vmatrix},$$

which equals $-\frac{1}{\mu_j \mu_k} R_{2j\,2k-1} = 0$ if $j<k$, but equals $+\frac{1}{\mu_j^2} R_{2j-1\,2j} = \frac{1}{\mu_j}$ if $j=k$. In what follows, let $j<k$. Then

$$C_{2j-1\,2k-1} = \frac{1}{\mu_j \mu_k} R_{2j\,2k} = 0, \quad C_{2j\,2k-1} = \frac{-1}{\mu_j \mu_k} R_{2j-1\,2k} = 0,$$

$$C_{2j\,2k} = \frac{1}{\mu_j \mu_k} R_{2j-1\,2k-1} = 0.$$

Corollary. From (4) it follows that, for transformation $A=(2)$,

$$\psi_{\xi,x} = \sum_{t=1}^{m} \frac{1}{\mu_t} \begin{vmatrix} \xi'_{2t-1} & \xi'_{2t} \\ x'_{2t-1} & x'_{2t} \end{vmatrix}. \quad (10)$$

4. Henceforth* we employ the standard notation

* The notation (α_{ij}) was advantageous in §§2-3. Contrast (1) and (6) with (11) and (12) as to symmetry. One advantage in the notation (11) lies in the form of the inverse.

$$S:\quad \xi_i' = \sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \qquad (i = 1, \ldots, m) \tag{11}$$

for a special Abelian transformation. In view of (6), its inverse is

$$S^{-1}:\quad \xi_i' = \sum_{j=1}^{m} (\delta_{ji}\xi_j - \gamma_{ji}\eta_j), \quad \eta_i' = \sum_{j=1}^{m} (-\beta_{ji}\xi_j + \alpha_{ji}\eta_j), \quad (i = 1, \ldots, m). \tag{12}$$

We shall usually designate S by its matrix. Thus for $m = 3$,

$$\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} & \alpha_{13} & \gamma_{13} \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} & \beta_{23} & \delta_{23} \\ \alpha_{31} & \gamma_{31} & \alpha_{32} & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ \beta_{31} & \delta_{31} & \beta_{32} & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}. \tag{13}$$

The symbols C_{rs} and R_{rs} being positional retain their meaning in the new notation. For a special Abelian transformation,

$$C_{2t-1\,2t} = R_{2t-1\,2t} = 1, \quad C_{rs} = R_{rs} = 0 \text{ (unless } s = r + 1 = \text{even).} \tag{14}$$

Formulae (76) and (78) of Linear Groups give the expanded form of (14).

Frequent use will be made of the following simple special Abelian transformations in the standard notation (unaltered variables being suppressed):

$$\begin{array}{ll} P_{ij} & = (\xi_i\xi_j)(\eta_i\eta_j); \quad M_i:\ \xi_i' = \eta_i, \quad \eta_i' = -\xi_i; \\ L_{i,\lambda}: & \xi_i' = \xi_i + \lambda\eta_i; \quad L'_{i,\lambda}:\ \eta_i' = \eta_i + \lambda\xi_i; \\ N_{i,j,\lambda}: & \xi_i' = \xi_i + \lambda\eta_j, \quad \xi_j' = \xi_j + \lambda\eta_i; \\ Q_{i,j,\lambda}: & \xi_i' = \xi_i + \lambda\xi_j, \quad \eta_j' = \eta_j - \lambda\eta_i; \\ R_{i,j,\lambda}: & \eta_i' = \eta_i - \lambda\xi_j, \quad \eta_j' = \eta_j - \lambda\xi_i; \\ T_{i,\lambda}: & \xi_i' = \lambda\xi_i, \qquad \eta_i' = \lambda^{-1}\eta_i. \end{array}$$

5. Lemma.* *In any field F, the characteristic equation*

$$\Delta(\rho) \equiv \begin{vmatrix} \alpha_{11} - \rho & \gamma_{11} & \alpha_{12} & \gamma_{12} & \cdots \\ \beta_{11} & \delta_{11} - \rho & \beta_{12} & \delta_{12} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{15}$$

of a special Abelian transformation S is a reciprocal equation.

From the form of the inverse (12) of S, we see that the first minors of α_{ij}, β_{ij}, γ_{ij}, δ_{ij} in $|S| \equiv \Delta(0)$ equal δ_{ij}, γ_{ij}, β_{ij}, α_{ij}, respectively. Since $\Delta(0) = 1$, it follows from the theory of determinants that any minor of order r whose diagonal elements all lie in the main diagonal of $|S|$ equals a similar minor of order $2m - r$. Hence if σ_r denotes the coefficient of $(-\rho)^{2m-r}$ in (15), $\sigma_r = \sigma_{2m-r}$.

Characteristic equation with a root in F, §§6–15.

6. Let S be a transformation of $G \equiv SA(2m, F)$ for which $\Delta(\rho) = 0$ has a root $\varkappa$ belonging to the given field F. Then there exists a linear homogeneous function ω with coefficients in F, not all zero, which S replaces by $\varkappa\omega$. Moreover, G contains a transformation V which replaces ξ_1 by ω (Linear Groups, top of p. 93). Then $V^{-1}SV \equiv S_1$ replaces ξ_1 by $\varkappa\xi_1$.

7. Suppose first that $\varkappa \neq \varkappa^{-1}$. There exists a function

$$\omega_1 = a_{11}\xi_1 + b_{11}\eta_1 + \cdots + a_{1m}\xi_m + b_{1m}\eta_m,$$

with coefficients in F and not reducing to $c\xi_1$, such that S_1 replaces ω_1 by $\varkappa^{-1}\omega_1$.

If $b_{11} \neq 0$, G contains a transformation U_1 which replaces ξ_1 by $b_{11}^{-1}\xi_1$, and η_1 by ω_1 (Linear Groups, p. 93). Then $U_1^{-1}S_1U_1 \equiv S_2$ replaces ξ_1 by $\varkappa\xi_1$ and η_1 by $\varkappa^{-1}\eta_1$. In view of the Abelian conditions, it has the form

$$\xi_1' = \varkappa\xi_1, \quad \eta_1' = \varkappa^{-1}\eta_1, \quad \xi_i' = \sum_{j=2}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=2}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j), \tag{16}$$
$$(i = 2, \ldots, m).$$

The problem is, therefore, reduced from m to $m - 1$ pairs of variables (§35, §37).

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* I first stated this theorem (with proof for $m = 2$) in the Transactions article cited in §1. A reviewer has suggested that it follows from the theory of bilinear alternating forms. But this theory does not seem to have been established for all fields, including those having a modulus. Note that for modulus 2, the expression of any form in terms of a symmetric and an alternating form is not valid.

If $b_{11}=0$, then a_{ij}, b_{ij} ($j=2, \ldots, m$) are not all zero. According as a_{ij} or b_{ij} is not zero, we transform S_1 by P_{2j} or $P_{2j}M_2$ and obtain a similar transformation S_1' with $a_{12} \neq 0$. By the successive generality theorem,* G contains a transformation

$$\xi_1' = \xi_1, \quad \eta_1' = \eta_1 - a_{11}a_{12}^{-1}\eta_2, \quad \xi_2' = \omega_1, \ldots \tag{17}$$

This is seen to transform S_1' into

$$\begin{pmatrix} \varkappa & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \beta_{11} & \varkappa^{-1} & \beta_{12} & 0 & \beta_{13} & \delta_{13} & \ldots & \beta_{1m} & \delta_{1m} \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ \beta_{21} & 0 & \beta_{22} & \varkappa & \beta_{23} & \delta_{23} & \ldots & \beta_{2m} & \delta_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{18}$$

To obtain a similar transformation (18′) with $\beta_{11}=\beta_{22}=0$, transform by $L'_{1,\tau}L'_{2,\sigma}$, where $\beta_{11}+\tau(\varkappa-\varkappa^{-1})=0$, $\beta_{22}+\sigma(\varkappa^{-1}-\varkappa)=0$.

Let first $m=2$. If $\beta_{12}=0$, then $\beta_{21}=0$ and (18′) is a transformation (16). If $\beta_{12}\neq 0$, $T_{2,\sigma}$, where $\sigma=\beta_{12}\varkappa$, transforms (18′) into

$$\xi_1' = \varkappa\xi_1, \quad \eta_1' = \varkappa^{-1}\eta_1 + \varkappa^{-1}\xi_2, \quad \xi_2' = \varkappa^{-1}\xi_2, \quad \eta_2' = \varkappa\eta_2 + \varkappa\xi_1. \tag{19}$$

Let next $m \geqq 3$. Deleting the first two rows and columns of (18′), we obtain a special Abelian transformation A on ξ_i, η_i ($i=2, \ldots, m$). Just as S_1 was conjugate with S_2 or (18′), so A is conjugate† with a transformation replacing ξ_2 by $\varkappa^{-1}\xi_2$, and η_2 by $\varkappa\eta_2$, or else with one replacing ξ_2 by $\varkappa^{-1}\xi_2$, and ξ_3 by $\varkappa\xi_3$. In the first case we transform by P_{12} and obtain B:

$$B = \begin{pmatrix} \varkappa^{-1} & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & \varkappa & \varkappa^2\beta_{12} & 0 & 0 & 0 & \ldots \\ 0 & 0 & \varkappa & 0 & 0 & 0 & \ldots \\ \beta_{12} & 0 & 0 & \varkappa^{-1} & \beta_{13} & \delta_{13} & \ldots \\ 0 & 0 & \alpha_{31} & 0 & \alpha_{33} & \gamma_{33} & \ldots \\ 0 & 0 & \beta_{31} & 0 & \beta_{33} & \delta_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

In the second case, we transform by $L'_{3,\rho}$ and make $\beta_{33}=0$, obtaining C:

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* American Journal, Vol. XXIII, p. 365.

† By means of a special Abelian transformation on ξ_i, η_i ($i > 1$).

$$C=\begin{pmatrix} \varkappa & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \varkappa^{-1} & \beta_{12} & 0 & \beta_{13} & \delta_{13} & \dots \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & 0 & \dots \\ \beta_{21} & 0 & 0 & \varkappa & \beta_{23} & 0 & \dots \\ \alpha_{31} & 0 & 0 & 0 & \varkappa & 0 & \dots \\ \beta_{31} & 0 & \beta_{32} & 0 & 0 & \varkappa^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In B the first and second coefficients in the unwritten rows are zero by the Abelian conditions. If $\beta_{12}=0$, B is of the form (16). If $\beta_{12}\neq 0$, we delete the first two rows and columns of B and obtain a special Abelian transformation B' on ξ_i, η_i $(i>1)$. There exists a function

$$\omega_2 = a_{22}\xi_2 + b_{22}\eta_2 + \dots + a_{2m}\xi_m + b_{2m}\eta_m,$$

with coefficients in F and not reducing to $l\xi_2$, such that B' replaces ω_2 by $\varkappa^{-1}\omega_2$. If $b_{22}\neq 0$, G contains a transformation U_2 which replaces $\xi_1, \eta_1, \xi_2, \eta_2$ by $\xi_1, \eta_1, b_{22}^{-1}\xi_2, \omega_2$, respectively. Then $U_2^{-1}BU_2$ is of the form B with $\beta_{1j}=\delta_{1j}=0$ $(j=3, \dots, m)$. Then $\alpha_{j1}=\beta_{j1}=0$ $(j=3, \dots, m)$ by the Abelian conditions. Transforming by $P_{12}\,T_{2,\sigma}$, where $\sigma=\beta_{12}\varkappa$, we obtain

$$S_4 = \text{Product of (19) by a transformation on } \xi_i, \eta_i \, (i=3, \dots, m). \tag{20}$$

The problem is therefore reduced from m to $m-2$ pairs of variables (see §38). Next, if $b_{22}=0$, we may take $a_{23}\neq 0$. Then G contains

$$\xi_1'=\xi_1,\ \eta_1'=\eta_1,\ \xi_2'=\xi_2,\ \eta_2'=\eta_2-a_{22}a_{23}^{-1}\eta_3,\ \xi_3'=\omega_2, \dots \tag{17'}$$

This transforms B into a transformation replacing ξ_3 by $\varkappa^{-1}\xi_3$, and otherwise of the form B. By Abelian conditions, $\delta_{33}=\varkappa$, $\delta_{13}=0$. Transforming by $L'_{3,\sigma}$, we make $\beta_{33}=0$. Then transforming by $Q_{1,3,\lambda}$, where $\beta_{13}-\lambda\beta_{12}=0$, we get

$$B_1=\begin{pmatrix} \varkappa^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \varkappa & \varkappa^2\beta_{12} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \varkappa & 0 & 0 & 0 & 0 & 0 & \dots \\ \beta_{12} & 0 & 0 & \varkappa^{-1} & 0 & 0 & \beta_{24} & \delta_{24} & \dots \\ 0 & 0 & 0 & 0 & \varkappa^{-1} & 0 & 0 & 0 & \dots \\ 0 & 0 & \beta_{31} & 0 & 0 & \varkappa & \beta_{34} & \delta_{34} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{21}$$

the first, second, fourth and sixth coefficients in the unwritten rows being zero. If* $m=3$, then $\beta_{31}=0$ and $P_{13}^{-1}B_1P_{13}$ is of the form (16).

We next discuss C for the case $m=3$. The Abelian conditions give

$$\beta_{31}=\beta_{13}\,,\quad \alpha_{31}=-\varkappa^2\delta_{13}\,,\quad \beta_{23}=\varkappa^2\beta_{32}\,,\quad \varkappa^{-1}\beta_{21}=\varkappa\beta_{12}-\delta_{13}\beta_{23}\,.$$

Now C is transformed into a similar transformation C' with $\beta_{31}=\beta_{13}=0$ by

$$L'_{1,\sigma}R_{1,3,\tau}\,,\quad \tau(\varkappa-\varkappa^{-1})=\beta_{13}\,,\quad \sigma(\varkappa-\varkappa^{-1})=\tau(\alpha_{31}-\delta_{13})\,.$$

If $\beta_{32}=0$, $P_{12}^{-1}C'P_{12}$ is of the form B. If $\beta_{32}\neq 0$, $Q_{3,1,\lambda}$, where $\beta_{12}-\lambda\beta_{32}=0$, transforms C' into a similar transformation C'' with also $\beta_{12}=0$. If $\delta_{13}=0$, then C'' is of the form (16). If $\delta_{13}\neq 0$, C'' is transformed into

$$\begin{pmatrix} \varkappa & 0 & 0 & 0 & 0 & 0 \\ 0 & \varkappa^{-1} & 0 & 0 & 0 & \varkappa^{-1} \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & 0 \\ -\varkappa & 0 & 0 & \varkappa & \varkappa & 0 \\ -\varkappa & 0 & 0 & 0 & \varkappa & 0 \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & \varkappa^{-1} \end{pmatrix} \qquad (22)$$

by $T_{2,\rho}\,T_{3,\sigma}$, where $\rho=\varkappa^2\delta_{13}\beta_{32}$, $\sigma=\varkappa^{-1}\delta_{13}^{-1}$.

We have now proved the preliminary

THEOREM.—*Within the group G, a transformation whose characteristic equation has a root $\varkappa\neq\varkappa^{-1}$ in the field F, is for $m=2$, conjugate with a type* (16) *or* (19); *for $m=3$, with a type* (16), (20) *or* (22).

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* If $m \geqq 4$, we delete the first four rows and columns of B_1 and obtain a special Abelian transformation B'_1 on ξ_i, η_i $(i=3,\ldots,m)$. As above, B_1 is conjugate with a like transformation with $\beta_{3j}=\delta_{3j}=0$ $(j=3,\ldots,m)$, or else with one replacing ξ_3 by $\kappa^{-1}\xi_3$, and ξ_4 by $\kappa\xi_4$. In the first case, $\beta_{31}=0$, so that its transform by P_{13} is of the form (16). In the second case, we transform by $L'_{4,\sigma}$ and make $\beta_{44}=0$, the resulting transformation being

$$C_1=\begin{pmatrix} \kappa^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0\;\ldots \\ 0 & \kappa & \kappa^2\beta_{12} & 0 & 0 & 0 & 0 & 0\;\ldots \\ 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0\;\ldots \\ \beta_{12} & 0 & 0 & \kappa^{-1} & 0 & 0 & \beta_{14} & \delta_{14}\;\ldots \\ 0 & 0 & 0 & 0 & \kappa^{-1} & 0 & 0 & 0\;\ldots \\ 0 & 0 & \beta_{32} & 0 & 0 & \kappa & \beta_{34} & \delta_{34}\;\ldots \\ 0 & 0 & \alpha_{42} & 0 & 0 & 0 & \kappa & 0\;\ldots \\ 0 & 0 & \beta_{42} & 0 & \beta_{43} & 0 & 0 & \kappa^{-1}\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

8. Let next $\Delta(\rho)=0$ have a root $\varkappa=\varkappa^{-1}$, but no root μ in F such that $\mu \neq \mu^{-1}$. By the general theory of canonical forms, there exists a linear function ω_1 with coefficients in F and not reducing to $c\xi_1$, such that S_1 replaces ω_1 by $\pm\omega+\tau\xi_1$. If $b_{11}\neq 0$, G contains a transformation U_1 such that $U_1^{-1}S_1U_1\equiv L$ has the form (see §7)

$$\left.\begin{aligned} &\xi_1'=\pm\xi_1, \quad \xi_i'=\sum_{j=2}^{m}(\alpha_{ij}\xi_j+\gamma_{ij}\eta_j),\\ &\qquad\qquad \eta_1'=\pm\eta_1+l\xi_1, \quad \eta_i'=\sum_{j=2}^{m}(\beta_{ij}\xi_j+\delta_{ij}\eta_j), \end{aligned}\right\}(i=2,\ldots,m), \tag{23}$$

which is studied in §39. If $b_{11}=0$, G contains a transformation (17) which transforms S_1 into R, where R replaces ξ_1 by $\pm\xi_1$, and ξ_2 by $\pm\xi_2+\tau\xi_1$. Deleting the first two rows and columns in R, we obtain a special Abelian transformation R' on $\xi_i, \eta_i\ (i>1)$, with $\xi_2'=\pm\xi_2$. Proceeding with R' as we did with S_1, we find that R is conjugate within G with one of the following two:

$$\begin{pmatrix} \pm1 & 0 & 0 & 0 & 0 & 0 & \ldots\\ \beta_{11} & \pm1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} & \ldots\\ \alpha_{21} & 0 & \pm1 & 0 & 0 & 0 & \ldots\\ \beta_{21} & 0 & \beta_{22} & \pm1 & 0 & 0 & \ldots\\ \alpha_{31} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} & \ldots\\ \beta_{31} & 0 & 0 & 0 & \beta_{33} & \delta_{33} & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \tag{24$_1$}$$

$$\begin{pmatrix} \pm1 & 0 & 0 & 0 & 0 & 0 & \ldots\\ \beta_{11} & \pm1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} & \ldots\\ \alpha_{21} & 0 & \pm1 & 0 & 0 & 0 & \ldots\\ \beta_{21} & 0 & \beta_{22} & \pm1 & \beta_{23} & \delta_{23} & \ldots\\ \alpha_{31} & 0 & \alpha_{32} & 0 & \pm1 & 0 & \ldots\\ \beta_{31} & 0 & \beta_{32} & 0 & \beta_{33} & \pm1 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{24$_2$}$$

9. Let first $m=2$, so that $(24)_1$ and $(24)_2$ are identical. If $\beta_{12}=\delta_{12}=0$, (24) is of the form (23). If $\delta_{12}=0$, $\beta_{12}\neq0$, the Abelian conditions give $\alpha_{21}=0$, $\beta_{21}=\beta_{12}$. If also $\beta_{11}\neq 0$, the transform of (24) by $Q_{1,2,\varkappa}$, where $\varkappa=\beta_{12}\beta_{11}^{-1}$, is of the form (23). If $\beta_{11}=0$, $\beta_{22}\neq 0$, we first transform (24) by P_{12} and obtain

a similar transformation with $\beta_{11} \neq 0$. If $\beta_{11} = 0$, $\beta_{22} = 0$, the transform of (24) by $T_{2,\beta_{12}}$ is M:

$$M = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 1 & 0 & 0 & \pm 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ -1 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

If F does not have modulus 2, Z transforms M into

$$\xi_1' = \pm \xi_1, \quad \eta_1' = \pm \eta_1 + \xi_1, \quad \xi_2' = \pm \xi_2, \quad \eta_2' = \pm \eta_2 - \xi_2,$$

of the form (23). For modulus 2, $M = R_{1,2,1}$ furnishes a new type. Finally, if $\delta_{12} \neq 0$, the transform of (24) by $T_{2,\delta_{12}^{-1}}$ is of like form with $\delta_{12} = 1$. We then transform by $L'_{2,\beta_{12}}$ and obtain R_1:

$$R_1 = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & 0 & 1 \\ -1 & 0 & \pm 1 & 0 \\ \mp \alpha & 0 & \alpha & \pm 1 \end{pmatrix}, \quad A_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -\alpha & 0 & \alpha & 1 \end{pmatrix}. \tag{25}$$

We assume first that F does not have modulus 2. Then $R_{1,2,\tau}$, where $\tau = -\frac{1}{2}\beta_{11}$, transforms R_1 into R_1', where R_1' is of the form R_1 with $\beta_{11} = 0$. For $\alpha = 0$, M_2 transforms R_1' into M. Let next $\alpha \neq 0$. For the upper signs, $R_1' = A_\alpha$. For the lower signs, $T_{1,-1}$ transforms R_1' into $A_{-\alpha}\, T_{1,-1}\, T_{2,-1}$. Now $T_{1,\kappa} T_{2,\kappa}$ transforms A_α into $A_{\alpha,\kappa^{-2}}$. Moreover, A_α and $A_{\alpha'}$ are not conjugate within $SA\,(4, F)$ if α'/α is a not-square. Indeed, $A_\alpha S = S A_{\alpha'}$ requires that

$$\gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = \alpha_{12} = \delta_{21} = 0, \quad \alpha_{22} = \alpha_{11}, \quad \delta_{22} = \delta_{11}, \quad \beta_{22} = \alpha\delta_{12},$$
$$\beta_{21} = -\beta_{12} - \alpha\delta_{12}, \quad \alpha\delta_{22} = \alpha'\alpha_{22}, \quad -\alpha'\alpha_{11} + \alpha'\alpha_{21} = -\beta_{22} - \alpha\delta_{22}.$$

Then Abelian condition R_{34} gives $\alpha_{22}\delta_{22} = 1$, whence $\alpha\delta_{22}^2 = \alpha'$. Moreover, A_α and $A_\beta T_{1,-1} T_{2,-1}$ are not conjugate since their characteristic equations have different roots.

Let next F be the $GF[2^n]$ or the infinite field $F^{(2)}$ defined as the aggregate of all the $GF[2^n]$, $n = 1, 2, 3, \ldots$. If $\alpha = 0$, the transform of R_1 by M_2 has the form (24) with $\delta_{12} = 0$, a case previously considered. If $\alpha \neq 0$, the transform of R_1 by $T_{1,\kappa} T_{2,\kappa}$ has the form R_1 with $\kappa^{-2}\alpha$ in place of α. Taking $\kappa = \alpha^{\frac{1}{2}}$, we obtain

$$R_\beta: \quad \xi_1' = \xi_1, \quad \eta_1' = \beta\xi_1 + \eta_1 + \eta_2, \quad \xi_2' = \xi_1 + \xi_2, \quad \eta_2' = \xi_1 + \xi_2 + \eta_2, \tag{26}$$

of period 4. Now $L'_{2,\tau}\,Q_{2,1,\tau}$ transforms R_β into R_b where

$$b=\beta+\tau+\tau^2. \tag{27}$$

For the field $F^{(2)}$, (27) has a root τ in the field, so that the R_β are all conjugate. Let next F be the $GF[2^n]$. Then τ can be determined in F to make $b=0$ if and only if β is a root of $f(\eta)=0$, where

$$f(\eta)\equiv\eta^{2^{n-1}}+\eta^{2^{n-2}}+\cdots\cdot+\eta^4+\eta^2+\eta. \tag{28}$$

Indeed the condition is $f(\beta)=f(\tau)+f(\tau^2)=\tau^{2^n}+\tau=0$ in the field. Hence the transformations R_β for which $f(\beta)=0$ are all conjugate within $SA(4,\,2^n)$. Likewise, the R_β for which $f(\beta)=1$ are all conjugate. Lastly, no transformation R_β with $f(\beta)=1$ is conjugate with R_0. For, if $R_\beta S=SR_0$, S must have the form

$$\begin{pmatrix}1&0&0&0\\ \beta_{11}&1&\beta_{12}&\beta_{22}\\ \beta_{22}&0&1&0\\ \beta_{21}&0&\beta_{22}&1\end{pmatrix}\begin{pmatrix}\beta_{21}=\beta_{12}+\beta_{22}+\beta\\ \beta_{21}=\beta_{12}+\beta_{22}^2\end{pmatrix}. \tag{29}$$

Hence must $\beta+\beta_{22}+\beta_{22}^2=0$ in the field, so that $f(\beta)=0$.

THEOREM.—*For $m=2$, (24) is conjugate with a type (23) or with*

R_0 or $R_{1,2,1}$, when F is the field $F^{(2)}$;

R_β or $R_{1,2,1}$, when F is the $GF[2^n]$,

where $\beta=0$ or a particular root of $f(\eta)=1$;

$$A_1,\ A_{\nu_i},\ A_1T_{1,-1}T_{2,-1},\ A_\nu\,T_{1,-1}T_{2,-1},$$

when F does not have modulus 2, where ν_i runs through a series of not-squares in F, such that no two have as their ratio a square, while every not-square has with some ν_i a ratio which is a square.

10. Let next $m=3$. Consider first $(24)_1$. Then

$$\begin{vmatrix}\alpha_{33}-\rho & \gamma_{33}\\ \beta_{33} & \delta_{33}-\rho\end{vmatrix}\equiv\rho^2-\rho\,(\alpha_{33}+\delta_{33})+1 \tag{Q}$$

is a factor of $\Delta(\rho)$. By hypothesis, it has no root $\varkappa$ in the field F, such that $\varkappa\neq\varkappa^{-1}$. If $\varkappa=\pm1$, we are led to $(24)_2$. If $\varkappa=\mp1$, F not having modulus 2, we transform by P_{13} and find, as in §8, that $(24)_1$ is conjugate with a type (23). There remains the case in which (Q) is irreducible in F, so that $\gamma_{33}\neq0$.

The transform of $(24)_1$ by $L'_{3,\lambda}$, where $\alpha_{33}-\lambda\gamma_{33}=0$, is of the form $(24)_1$ with $\alpha_{33}=0$:

$$\begin{pmatrix} \pm 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \pm 1 & 0 & 0 & 0 \\ \beta_{21} & 0 & \beta_{22} & \pm 1 & 0 & 0 \\ \alpha_{31} & 0 & 0 & 0 & 0 & \gamma \\ \beta_{31} & 0 & 0 & 0 & -\gamma^{-1} & d \end{pmatrix}. \qquad (24)$$

Abelian condition R_{25} gives $\alpha_{31}=\pm\gamma\beta_{13}$. Transforming by $Q_{3,1,-\gamma\beta_{13}}$, we obtain a transformation (24′) with $\beta_{13}=0$, $\alpha_{31}=0$. We next show that we can take $\delta_{13}=0$. If $\delta_{13}\neq 0$, we transform by $T_{1,\delta_{13}}$ and obtain a type (24′) with $\delta_{13}=1$, and $\beta_{13}=\alpha_{31}=0$ as before. Transforming the latter by

$$Q_{3,1,\pm\gamma a}R_{3,1,-a}, \qquad a(d\mp 2)=\pm\gamma^{-1},$$

we obtain a transformation differing from it only in the coefficients β_{11} and δ_{13}, and having $\delta_{13}=0$. Note that a is determined in F since $d\mp 2\neq 0$ in view of the irreducibility of (Q), which is now $\rho^2-d\rho+1$.

There results (24′) with $\beta_{13}=\delta_{13}=0$. It is thus a product of the type treated in §9 by a transformation S_γ on ξ_3, η_3, treated in §1. Let F be the $GF[p^n]$. Then S_γ is conjugate with S_1 under transformation of determinant unity on ξ_3, η_3 within F. *Hence, for the* $GF[p^n]$, *the new canonical forms furnished by* $(24)_1$ *are* PS_1, *where* $P=R_\beta$ *or* $R_{1,2,1}$ *if* $p=2$, $P=A_1$, A_ν, $A_1T_{1,-1}T_{2,-1}$ *or* $A_\nu T_{1,-1}T_{2,-1}$ *if* $p>2$.

We may give to S_1 the ultimate canonical form $T_{3,\kappa}$, where $\kappa^{p^n+1}=1$, κ belonging therefore to the $GF[p^{2n}]$.

Consider next the type $(24)_2$. Denote by R' the special Abelian transformation obtained from it by deleting the first two rows and columns. We may give to R' one of the canonical forms of §9 on ξ_2, η_2, ξ_3, η_3, upon applying a transformation of the latter variables. When R' is of the type (23), $(24)_2$ becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ -\delta_{12} & 0 & 1 & 0 & 0 & 0 \\ \beta_{21} & 0 & \beta_{22} & 1 & 0 & 0 \\ -\delta_{13} & 0 & 0 & 0 & 1 & 0 \\ \beta_{31} & 0 & 0 & 0 & \beta_{33} & 1 \end{pmatrix}, \qquad \begin{aligned} \beta_{21}&=\beta_{12}-\delta_{12}\beta_{22}, \\ \beta_{31}&=\beta_{13}-\delta_{13}\beta_{33}, \end{aligned} \qquad (30)$$

or its product by $T \equiv T_{1,-1} T_{2,-1} T_{3,-1}$, which changes the sign of each variable. Let first F have modulus 2. For $R' = R_\beta$ $(24)_2$ becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ -\delta_{12} & 0 & 1 & 0 & 0 & 0 \\ \beta_{21} & 0 & \beta & 1 & 0 & 1 \\ \delta_{12}-\delta_{13} & 0 & -1 & 0 & 1 & 0 \\ \beta_{31} & 0 & -\alpha & 0 & \alpha & 1 \end{pmatrix}, \quad \begin{array}{l} \beta_{21} = \beta_{12} + \beta_{13} - \beta\delta_{12}, \\ \beta_{31} = \beta_{13} + \alpha\delta_{12} - \alpha\delta_{13}, \end{array} \tag{31}$$

where $\alpha = 1$. For $R' = M_3^{-1} R_{2,3,1} M_3$, $(24)_2$ becomes (31) for $\beta = 0$, $\alpha = 0$. If F does not have modulus 2, we may take $R' = A_\alpha T_{1,\pm 1} T_{2,\pm 1}$, $\alpha = 1$ or ν_i. Then $(24)_2$ becomes (31) for $\beta = 0$, or its product by T.

11. Consider transformation (30). If $\delta_{12} \neq 0$, its transform by $R_{2,3,\lambda}$, where $\beta_{31} + \lambda\delta_{12} = 0$, differs from (30) only in β_{12}, β_{13}, β_{21}, β_{31}, with β_{31} now zero. When the result is transformed by $L'_{2,\sigma}$, $\beta_{21} - \sigma\delta_{12} = 0$, only β_{12} and β_{21} are altered, with $\beta_{21} = 0$. If also $\delta_{13} = 0$, we transform by P_{13} and obtain a transformation (23). Next, if $\delta_{12} = 0$, $\delta_{13} \neq 0$, we transform (30) by $R_{2,3,\lambda}$, where $\beta_{31} + \lambda\delta_{13} = 0$, and obtain a transformation (30) with $\delta_{12} = \beta_{31} = 0$, which leads to a (23). Hence there remain two cases:

$$\delta_{12} \neq 0,\ \delta_{13} \neq 0,\ \beta_{21} = 0,\ \beta_{31} = 0;\ \delta_{12} = 0,\ \delta_{13} = 0,\ \beta_{21} \neq 0,\ \beta_{31} \neq 0.$$

If, in the second case, $\beta_{22} \neq 0$, we transform by $Q_{2,1,\lambda}$, $\beta_{12} - \lambda\beta_{22} = 0$, and obtain a similar transformation with $\beta_{12} = \beta_{21} = 0$. Likewise, if $\beta_{33} \neq 0$, we transform by $Q_{3,1,\lambda}$, $\beta_{13} - \lambda\beta_{33} = 0$, and have $\beta_{13} = \beta_{31} = 0$. But, if $\beta_{22} = \beta_{33} = 0$, it is transformed by $Q_{3,2,\lambda}$, $\beta_{21} - \lambda\beta_{31} = 0$, into a similar transformation with $\beta_{12} = \beta_{21} = 0$. Hence in each of these three sub-cases, we obtain a transformation (23). For the first main case, the Abelian conditions give

$$\beta_{12} = \delta_{12}\beta_{22}, \quad \beta_{13} = \delta_{13}\beta_{33}.$$

Transforming by $T_{2,\delta_{12}^{-1}} T_{3,\delta_{13}^{-1}}$, we obtain*

$$S_{b_1, b_2, b_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 1 & b_2 & 1 & b_3 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_3 & 1 \end{pmatrix}. \tag{32}$$

* For properties of (32) see Amer. Math. Monthly, 1904, p. 88.

If $b_2 = b_3 = 0$, the transform of (32) by $Q_{3,2,-1}$ leaves ξ_3 and η_3 unaltered, and hence leads to (23). Since P_{23} transforms S_{b_1, b_2, b_3} into S_{b_1, b_3, b_2}, we may assume that $b_3 \neq 0$.

We seek the conditions under which (32) is conjugate within G with a transformation (23). In view of the successive generality theorem,* this conjugacy will follow if, and only if, new variables

$$X = a\xi_1 + b\eta_1 + c\xi_2 + d\eta_2 + e\xi_3 + j\eta_3, \qquad Y = A\xi_1 + B\eta_1 + C\xi_2 + D\eta_2 + E\xi_3 + J\eta_3 \tag{33}$$

can be determined such that

$$\begin{vmatrix} a & b \\ A & B \end{vmatrix} + \begin{vmatrix} c & d \\ C & D \end{vmatrix} + \begin{vmatrix} e & j \\ E & J \end{vmatrix} = 1, \tag{34}$$

and such that, under transformation (32),

$$X' = X, \quad Y' = Y + tX, \tag{35}$$

where $a, A, \ldots, J, t$ all belong to the field F. The conditions for (35) are

$$b = 0, \quad j = 0, \quad db_2 = 0, \quad c + e = 0,$$
$$B = 0, \quad J = te/b_3, \quad td = 0, \quad Db_2 = tc, \quad C + E = -ta.$$

If $b_2 \neq 0$, condition (17) becomes

$$tc^2(b_2^{-1} + b_3^{-1}) = 1,$$

whence $c \neq 0$, $b_2 + b_3 \neq 0$. Inversely, if $b_2 + b_3 \neq 0$, all the conditions can be satisfied. For example, we may take

$$X = \xi_2 - \xi_3, \quad Y = (b_3\eta_2 - b_2\eta_3)/(b_2 + b_3), \quad t = b_2 b_3/(b_2 + b_3).$$

If $b_2 = 0$, the conditions reduce to

$$b = c = B = J = t = 0, \quad c + e = 0, \quad C + E = 0, \quad cD - dC = 1.$$

It suffices to take $c = D = 1$, $d = C = 0$, $a = A = 0$, whence $X = \xi_2 - \xi_3$, $Y = \eta_2$.

THEOREM.—*S_{b_1, b_2, b_3} is conjugate within G with* (23) *unless* $b_3 = -b_2 \neq 0$.

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* American Journal, Vol. XXIII, p. 365.

Consider next $S_{b_1, b_2, -b_2}$, where $b_2 \neq 0$. Now $T_{1, \kappa^{-1}} T_{2, \kappa^{-1}} T_{3, \kappa^{-1}}$ transforms S_{b_1, b_2, b_3} into $S_{\kappa^2 b_1, \kappa^2 b_2, \kappa^2 b_3}$. Hence if F be the $GF[2^n]$ or $F^{(2)}$, so that every element is a square, $S_{b_1, b_2, -b_2}$ is conjugate with a certain $S_{b, 1, 1}$. To obtain a like result when F does not have modulus 2, we note first that $R_{1, 2, \lambda}$ transforms S_{b_1, b_2, b_3} into $S_{b_1 + 2\lambda, b_2, b_3}$, so that we may give to b_1 an arbitrary value in F. Whether F has modulus 2 or not, the product

$$Q_{2, 3, -1} P_{12} N_{2, 3, b_1^{-1}} M_3 T_{2, -b_1 b_2^{-1}} T_{3, -b_2^{-1}} R_{2, 3, -\lambda} L'_{2, -2\lambda},$$

where $\lambda = b_2^2 / b_1$, $b_1 \neq 0$, transforms $S_{b_1, b_2, -b_2}$ into $S_{b_2, \lambda, -\lambda}$. By choice of b_1, we may make $\lambda = 1$. Hence for any F, $S_{b_1, b_2, -b_2}$ is conjugate with $S_{b, 1, -1}$, where $b = 0$ if F does not have modulus 2.

If $S_{b, 1, -1} S = S S_{c, 1, -1}$, then S is found to have the form

$$\begin{pmatrix} \alpha_{11} & 0 & -\gamma & 0 & \gamma & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \beta_{22} + \beta_{32} - c\gamma & \beta_{13} & -\beta_{23} - \beta_{33} - c\gamma \\ b\gamma - \beta_{22} - \beta_{23} & 0 & \alpha & \gamma & \alpha_{11} - \alpha & \gamma \\ \beta_{21} & \gamma & \beta_{22} & \alpha - \gamma & \beta_{23} & \alpha - \alpha_{11} - \gamma \\ b\gamma + \beta_{32} + \beta_{33} & 0 & \alpha - \delta_{11} & \gamma & \alpha_{11} + \delta_{11} - \alpha & \gamma \\ \beta_{31} & -\gamma & \beta_{32} & \delta_{11} - \alpha + \gamma & \beta_{33} & \alpha_{11} + \delta_{11} - \alpha + \gamma \end{pmatrix}, \tag{36}$$

$$\beta_{12} + \beta_{21} + \beta_{13} + \beta_{31} - \beta_{22} - \beta_{23} - \beta_{32} - \beta_{33} = b\delta_{11} - c\alpha_{11}. \tag{37}$$

The Abelian conditions are $R_{1j} \equiv 0 \; (j = 3, 4, 5, 6)$, $R_{35} \equiv 0$,

$$\begin{aligned} R_{12} \equiv {} & \alpha_{11}\delta_{11} - \gamma(\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}) = 1, \\ R_{23} \equiv {} & \gamma\beta_{12} + \gamma\beta_{13} - \alpha\beta_{22} - \alpha\beta_{32} + \delta_{11}\beta_{22} \\ & \qquad + \delta_{11}\beta_{23} + (\alpha_{11} - \alpha)(\beta_{23} + \beta_{33}) - b\gamma\delta_{11} + c\gamma\alpha_{11} = 0, \\ R_{24} \equiv {} & \gamma\beta_{11} - \delta_{11}\beta_{21} + (\alpha - \gamma)\beta_{12} + (\alpha - \alpha_{11} - \gamma)\beta_{13} \\ & \qquad - \beta_{22}(\beta_{22} + \beta_{32} - c\gamma) + \beta_{23}(\beta_{23} + \beta_{33} + c\gamma) = 0, \\ R_{34} \equiv {} & b\gamma^2 - \alpha_{11}\gamma - \alpha_{11}^2 + 2\alpha\alpha_{11} - 2\gamma\beta_{22} - 2\gamma\beta_{23} = 1, \\ R_{46} \equiv {} & - \gamma\beta_{21} - \gamma\beta_{31} + (\delta_{11} - \alpha + \gamma)\beta_{22} + (\gamma - \alpha)\beta_{32} \\ & \qquad + (\alpha_{11} + \delta_{11} - \alpha + \gamma)\beta_{23} + (\alpha_{11} - \alpha + \gamma)\beta_{33} = 0. \end{aligned}$$

Next, $R_{25} = 0$ reduces to $R_{12} = 0$, while $R_{26} + R_{24}$ gives

$$\begin{aligned} \delta_{11}(\beta_{12} + \beta_{13} - \beta_{21} - \beta_{31}) + c\gamma(\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}) & \\ - (\beta_{22} + \beta_{32})^2 + (\beta_{23} + \beta_{33})^2 = 0. & \end{aligned} \tag{38}$$

Further, $R_{36} + R_{34} = R_{12}$, $R_{56} + R_{34} = 2R_{12}$, $R_{23} - R_{46}$ equals γ times the left

member of (37). Hence the conditions on (36) reduce to (37), (38) and those marked R_{13}, R_{24}, R_{34}, R_{46}.

Let F have modulus 2 and set $B \equiv \beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}$. Then (37), R_{12}, R_{34} become

$$\beta_{12} + \beta_{21} + \beta_{13} + \beta_{31} = B + b\delta_{11} - c\alpha_{11}, \quad \alpha_{11}\delta_{11} - \gamma B = 1, \quad b\gamma^2 - \alpha_{11}\gamma - \alpha_{11}^2 = 1. \quad (39)$$

In view of these R_{46} and (38) become

$$\gamma(\beta_{21} + \beta_{31}) + B(\gamma + \alpha) + \delta_{11}(\beta_{22} + \beta_{23}) + \alpha_{11}(\beta_{23} + \beta_{33}) = 0, \quad B^2 + B\delta_{11} + b\delta_{11}^2 = c. \quad (40)$$

Hence, the conditions on (36) reduce to R_{24}, (39), (40). Let first $b \neq 0, c = 0$. Then $\delta_{11} \neq 0$ by $(39)_2$ and $(40)_2$, so that the latter may be written

$$(B/\delta_{11})^2 + B/\delta_{11} = b. \quad (41)$$

If F be the field $F^{(2)}$, (41) can always be solved in F. If F be the $GF[2^n]$, it can be solved in F if, and only if, $f(b) = 0$, f being defined by (28). Suppose this condition satisfied. Then, in either case, conditions R_{24}, (39) and (40), are all satisfied by

$$\alpha_{11} = \delta_{11} = 1, \quad \gamma = 0, \quad \alpha = 1, \quad \beta_{12} = \beta_{21} = \beta_{22} = \beta_{23} = \beta_{31} = \beta_{32} = 0, \quad \beta_{13} = B^2, \quad \beta_{33} = B. \quad (42)$$

Let next $f(b) = f(c) = 1$. Then $B^2 + B = b + c$ can be solved in F. For any such value of B and for the values (42), conditions R_{24}, (39), (40) are all satisfied.

THEOREM.—*If F does not have modulus 2, or if F is $F^{(2)}$, every $S_{b_1, b_2, -b_2}$ is conjugate with $S_{0,1,-1}$ within G. If F is the $GF[2^n]$, the $S_{b_1, b_2, -b_2}$ are conjugate with $S_{0,1,-1}$ or $S_{b,1,-1}$, b being a particular root of $f(\eta) = 1$, while the latter two are not conjugate.*

12. Consider next transformation (31). It is transformed by $Q_{2,1,\delta_{13}}$ into a similar transformation with $\delta_{13} = 0$. If $\alpha \neq 0$, the latter is transformed by $Q_{3,1,\lambda}$, $\beta_{13} - \lambda\alpha = 0$, into a similar transformation with $\beta_{13} = \delta_{13} = 0$. Now $R_{1,3,\lambda}$ transforms (31) into a transformation differing from it only in the coefficients β_{11}, β_{12}, β_{21}, having $\beta_{12} + \lambda$ in place of β_{12}, and $\beta_{21} + \lambda$ in place of β_{21}. Hence, if $\alpha \neq 0$, we may take $\beta_{13} = \delta_{13} = \beta_{12} = 0$; if $\alpha = 0$, we may take $\delta_{13} = \beta_{12} = 0$.

Consider, first, the case $\alpha \neq 0$, $\beta_{13} = \delta_{13} = \beta_{12} = 0$. If $\delta_{12} = 0$, (31) is of the form (23). If $\delta_{12} \neq 0$, $T_{1,\delta_{12}}$ transforms (31) into

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta' & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\beta & 0 & \beta & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ \alpha & 0 & -\alpha & 0 & \alpha & 1 \end{pmatrix}. \tag{43}$$

To show that (43) is not conjugate within G with a transformation (23), we proceed as in §11. Let X and Y be defined by (33). The conditions for (35) are now

$$b = c = d = e = j = 0, \quad B = D = E = J = 0, \quad C = -ta,$$

so that (34) is impossible.

The subdivisions of the next sections are based on §10.

13. Consider (43) for $\beta = 0$, $\alpha \neq 0$, F not having modulus 2. We then denote (43) by $S_{\beta',\alpha}$. If S is the general transformation (13), $S_{\beta',\alpha} S = S S_{B,A}$ requires

$$\alpha_{12} = \alpha_{13} = \alpha_{23} = \delta_{21} = \delta_{31} = \delta_{32} = 0, \quad \delta_{11} = \delta_{22} = \delta_{33}, \quad \alpha_{11} = \alpha_{22} = \alpha_{33},$$
$$\alpha_{32} = \alpha_{21}, \quad \delta_{23} = \delta_{12}, \quad \alpha\delta_{11} = A\alpha_{11}, \quad \beta_{33} = \alpha\delta_{12}, \quad \gamma_{ij} = 0 \ (i, j, = 1, 2, 3),$$
$$\beta_{23} = \alpha\delta_{13}, \quad \beta_{32} = -A\alpha_{31}, \quad \beta_{33} = -A\alpha_{21},$$
$$\beta_{22} = -\beta_{13} - \alpha\delta_{13}, \quad \beta_{32} = -\beta_{23} - \alpha\delta_{12},$$
$$\beta_{31} = \beta_{23} - \beta_{22} + \alpha\delta_{12}, \quad \beta'\delta_{11} - \beta_{12} + \beta_{13} + \alpha\delta_{13} = B\alpha_{11} + \beta_{21}.$$

Hence S must be of the form

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{11} & 0 & 0 & 0 \\ \beta_{21} & 0 & -\beta_{13} - \alpha\delta_{13} & \delta_{11} & \alpha\delta_{13} & \delta_{12} \\ \alpha_{31} & 0 & \alpha_{21} & 0 & \alpha_{11} & 0 \\ \beta_{31} & 0 & -A\alpha_{31} & 0 & \alpha\delta_{12} & \delta_{11} \end{pmatrix}, \tag{44}$$

$$A\alpha_{11} = \alpha\delta_{11}, \quad A\alpha_{21} = -\alpha\delta_{12}, \quad A\alpha_{31} = \alpha\delta_{12} + \alpha\delta_{13}, \tag{45}$$
$$\beta_{31} = \beta_{13} + 2\alpha\delta_{13} + \alpha\delta_{12}, \quad \beta_{21} = \beta'\delta_{11} - B\alpha_{11} + \alpha\delta_{13} - \beta_{12} + \beta_{13}. \tag{46}$$

The conditions that (44) shall be Abelian then reduce to

$$\alpha_{11}\delta_{11}=1, \quad \delta_{11}\delta_{12}+2\delta_{11}\delta_{13}-\delta_{12}^2=0, \tag{47}$$

$$\alpha\delta_{12}\delta_{13}-\alpha\delta_{11}\delta_{13}-\alpha\delta_{13}^2+2\delta_{11}\beta_{12}+2\delta_{12}\beta_{13}-\delta_{11}\beta_{13}-\beta'\delta_{11}^2+B=0. \tag{48}$$

Hence A/α must equal a square δ_{11}^2 in the field F. If this condition be satisfied, $S_{\beta',\alpha}$ is conjugate with $S_{B,A}$ within G. Indeed, we may employ (44) as transformer, where

$$\alpha_{11}=\delta_{11}^{-1}, \quad \beta_{11}=0, \quad \delta_{12}=\delta_{13}=\alpha_{21}=\alpha_{31}=\beta_{12}=\beta_{21}=0,$$
$$\beta_{31}=\beta_{13}=\alpha_{11}B-\delta_{11}\beta'.$$

Hence every $S_{\beta',\alpha}$ is conjugate within G with $S_{0,1}$ or one of the S_{0,ν_i}.

14. Consider (43) for $\alpha=1$, F having modulus 2. Transforming it by $\xi_3'=\xi_3-\eta_3$, then by $Q_{1,2,-1}$, and finally by $\eta_1'=\eta_1+\beta'\xi_1$, we get

$$W_{\delta,\beta}: \left\{\begin{matrix} \xi_1'=\xi_2, & \eta_1'=\delta\xi_1+\eta_2, & \xi_2'=\xi_1, \\ \eta_2'=\beta\xi_1+\eta_1+\delta\xi_2+\eta_3, & \xi_3'=\eta_3, & \eta_3'=\xi_1+\xi_3. \end{matrix}\right\} \tag{49}$$

If $W_{\delta,\beta}S=SW_{D,B}$, S must have the form

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \alpha_{12} & \beta_{13} & \delta_{13} \\ \alpha_{12} & 0 & \alpha_{11} & 0 & 0 & 0 \\ \beta_{21} & \alpha_{12} & \beta_{22} & \alpha_{11} & \delta_{13} & \alpha_{12}+\beta_{13} \\ \alpha_{31} & 0 & \alpha_{12}+\beta_{31} & 0 & \alpha_{11} & \alpha_{12} \\ \beta_{31} & 0 & \alpha_{31} & 0 & \alpha_{12} & \alpha_{11} \end{pmatrix},$$

$$\left.\begin{aligned} \beta_{22}&=\beta_{11}+D\alpha_{12}+\delta\alpha_{12}, \\ \beta_{21}&=\delta_{13}+\beta_{12}+\beta\alpha_{12}+D\alpha_{11}+\delta\alpha_{11}, \\ \beta_{31}&=\beta_{13}+\alpha_{12}+B\alpha_{11}+\beta\alpha_{11}, \\ \alpha_{31}&=\delta_{13}+B\alpha_{12}+\beta\alpha_{12}, \end{aligned}\right\} \tag{50}$$

The Abelian conditions on (50) reduce to

$$\alpha_{11}+\alpha_{12}=1, \quad \alpha_{11}^2+\alpha_{11}+B+\beta=0, \tag{51}$$

$$\delta_{13}^2+\alpha_{11}\delta_{13}+\beta_{13}^2+\alpha_{12}\beta_{13}+D+\delta+\alpha_{11}\alpha_{12}\beta=0. \tag{52}$$

If F be the field $F^{(2)}$, $(51)_2$ can be solved for α_{11} in F. If F be the $GF[2^n]$, the

condition for solution is $f(B)=f(\beta)$. Suppose this condition satisfied. In either case, (52) can be solved in F. For example, we may take*

$$\beta_{13}=\delta_{13}=D+\delta+\alpha_{11}\alpha_{12}\beta. \tag{53}$$

THEOREM.—*For the field $F^{(2)}$, every $W_{\delta,\beta}$ is conjugate with $W_{0,0}$. For the $GF[2^n]$, $W_{\delta,\beta}$ is conjugate with $W_{0,0}$ or $W_{0,b}$, where b is a particular root of $f(\eta)=1$, the two being not conjugate within G.*

15. It remains to consider (31) for $\alpha=\beta=0$, F having modulus 2. By §12, we may take $\beta_{13}=\delta_{13}=0$. Then (31) may be written

$$V_{\gamma,\delta,\epsilon}:\left\{\begin{matrix}\xi_1'=\xi_1, & \eta_1'=\gamma\xi_1+\eta_1+\delta\eta_2+\epsilon\xi_3, & \xi_2'=\delta\xi_1+\xi_2 \\ \eta_2'=\epsilon\xi_1+\eta_2+\eta_3, & \xi_3'=\delta\xi_1+\xi_2+\xi_3, & \eta_3'=\epsilon\xi_1+\eta_3.\end{matrix}\right\} \tag{54}$$

To test its conjugacy with (23), we proceed as in §11. If $\delta=\epsilon=0$, V is of the form (23). Let next δ and ϵ not both $=0$. Then (35) requires

$$b=d=e=0,\quad c\delta+j\epsilon=0,\quad B=0,$$
$$E=tc,\quad D=tj,\quad C\delta+D\epsilon+E\delta+J\epsilon=ta.$$

Then (34) becomes $1=cD-jE=0$ and is impossible, so that the two are not conjugate within the Abelian group.

Since $P_{23}M_2M_3$ transforms $V_{\gamma,\delta,\epsilon}$ into $V_{\gamma,\epsilon,\delta}$, we may assume that $\delta\neq 0$. Then the Abelian transformation

$$\xi_1'=\xi_1,\quad \eta_1'=\eta_1+\gamma/\delta\xi_3,\quad \xi_2'=1/\delta\xi_2,$$
$$\eta_2'=\gamma/\delta\xi_2+\delta\eta_2+\epsilon\xi_3,\quad \xi_3'=1/\delta\xi_3,\quad \eta_3'=\gamma\xi_1+\epsilon\xi_2+\delta\eta_3,$$

transforms $V_{\gamma,\delta,\epsilon}$ into $V_{0,1,0}$. Further, the Abelian transformation

$$\xi_1'=\xi_1+\eta_3,\quad \eta_1'=\xi_3,\quad \xi_2'=\xi_1+\xi_2+\eta_2,$$
$$\eta_2'=\eta_1+\xi_3,\quad \xi_3'=\xi_2+\eta_2,\quad \eta_3'=\eta_1+\xi_2+\xi_3$$

transforms $V_{0,1,0}$ into $S_{0,1,1}$.

Characteristic equation with no root in F, §§16–33.

16. THEOREM.—*Within $G\equiv SA(2m,F)$, any transformation S whose characteristic equation has no root in F, is conjugate with a transformation which replaces ξ_1 by $\gamma\eta_1$.*

* As a check, I verified that, when (51) and (53) hold and $\beta_{11}=\beta_{12}=0$, transformation (50) is Abelian and transforms $W_{\delta,\beta}$ into $W_{D,B}$.

Let S replace ξ_1 by $\omega \equiv \alpha_{11}\xi_1 + \gamma_{11}\eta_1 + \ldots.$ If $\gamma_{11} \neq 0$, there exists in G a transformation V which replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by ω. Then $V^{-1}SV$ replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$. The same result follows if S is conjugate within G with a transformation having $\gamma_{11} \neq 0$. Suppose then that $\gamma_{11} = 0$ in all the conjugates to S. The transform of S by M_1, P_{1i} or $P_{1i}M_1$, where $i > 1$, has β_{11}, γ_{ii} or β_{ii} respectively, as the coefficient of η_1 in ξ_1'. Hence, every $\beta_{ii} = \gamma_{ii} = 0$ $(i = 1, \ldots, m)$. The transform of the resulting transformation S' by $\eta_i' = \eta_i + \eta_j$, $\xi_j' = \xi_j - \xi_i$ has $\beta_{ij} + \beta_{ji}$ as the coefficient of ξ_i in η_i', and $-\gamma_{ij} - \gamma_{ji}$ as the coefficient of η_j in ξ_j'. Hence

$$\beta_{ij} = -\beta_{ji}, \quad \gamma_{ij} = -\gamma_{ji}, \qquad (i, j = 1, \ldots, m;\ i \neq j).$$

The transform of S' by $\xi_i' = \xi_i + \eta_j$, $\xi_j' = \xi_j + \eta_i$ has $\delta_{ji} - \alpha_{ij}$ as the coefficient of η_i in ξ_i'. Hence $\alpha_{ij} = \delta_{ji}$ for $i \neq j$. Finally, the transform of S' by $L_{i,1}$ has $\delta_{ii} - \alpha_{ii}$ as the coefficient of η_i and ξ_i'. Hence $\alpha_{ii} = \delta_{ii}$. It follows from §4 that S' is of period 2. By the general theory of canonical forms, the roots of the characteristic equation of S' all satisfy $x^2 = 1$, contrary to hypothesis.

17. Let S be a transformation of G which replaces ξ_1 by $\gamma\eta_1$, and for which $\Delta(\rho) = 0$ has no root in F. Thus S replaces η_1 by

$$-\gamma^{-1}\xi_1 + \delta\eta_1 + \tau, \quad \tau \equiv \beta_{12}\xi_2 + \delta_{12}\eta_2 + \ldots + \beta_{1m}\xi_m + \delta_{1m}\eta_m.$$

If $\tau \equiv 0$, S becomes, in view of the Abelian conditions

$$\left.\begin{aligned} &\xi_1' = \gamma\eta_1, \quad \xi_i' = \sum_{j=2}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ &\qquad \eta_1' = -\gamma^{-1}\xi + \delta\eta_1, \quad \eta_i' = \sum_{j=2}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j), \end{aligned}\right\} \quad (i = 2, \ldots, m,) \qquad (55)$$

so that the problem is reduced from m to $m - 1$ pairs of variables (see §40). If $\tau \not\equiv 0$, there exists a transformation Σ on ξ_i, $\eta_i (i > 1)$, which replaces ξ_2 by τ. Then $\Sigma^{-1}S\Sigma = S_1$ replaces ξ_1 by $\gamma\eta_1$ and η_1 by $-\gamma^{-1}\xi_1 + \delta\eta_1 + \xi_2$.

18. For $m = 2$, S_1 becomes, in view of the Abelian conditions,

$$S_1 = \begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 \\ 0 & \gamma\gamma_{22} & \alpha_{22} & \gamma_{22} \\ 0 & \gamma\delta_{22} & \beta_{22} & \delta_{22} \end{pmatrix}, \quad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

If $\gamma_{22}=0$, the characteristic equation of S_1 has the root α_{22} in the field F, contrary to hypothesis. Hence $\gamma_{22}\neq 0$. Then the transform of S_1 by $\eta_2'=\eta_2-\lambda\xi_2$, where $\delta_{22}+\lambda\gamma_{22}=0$, has the form

$$Q=\begin{pmatrix} 0 & \gamma_1 & 0 & 0 \\ -\gamma_1^{-1} & \delta & 1 & 0 \\ 0 & \gamma_1\gamma & \alpha & \gamma \\ 0 & 0 & -\gamma^{-1} & 0 \end{pmatrix}. \tag{56}$$

Its characteristic determinant equals

$$\rho^4-\rho^3(\alpha+\delta)+\rho^2(2+\alpha\delta-\gamma_1\gamma)-\rho(\alpha+\delta)+1. \tag{57}$$

The latter has the factor $\rho^2-\sigma\rho+1$ if and only if

$$\sigma^2-\sigma(\alpha+\delta)+\alpha\delta-\gamma_1\gamma=0. \tag{58}$$

Assuming here that (58) has a root σ in the field F and that $\rho^2-\sigma\rho+1$ is irreducible in F, we determine the conditions under which Q is conjugate with a transformation (55). Let

$$X=a\xi_1+b\eta_1+c\xi_2+d\eta_2,\quad Y=A\xi_1+B\eta_1+C\xi_2+D\eta_2, \tag{33'}$$

where $a, b, \ldots . D$, and τ (below) are elements of F such that

$$\begin{vmatrix} a & b \\ A & B \end{vmatrix}+\begin{vmatrix} c & d \\ C & D \end{vmatrix}=1. \tag{34'}$$

The conditions that Q shall replace X by τY, and Y by $-\tau^{-1}X+\sigma Y$, are

$$\tau A=-b\gamma_1^{-1},\quad \tau B=a\gamma_1+b\delta+c\gamma_1\gamma,\quad \tau C=b+c\alpha-d\gamma^{-1},\quad \tau D=c\gamma,$$
$$\sigma A-\tau^{-1}a=-B\gamma_1^{-1},\quad \sigma B-\tau^{-1}b=A\gamma_1+B\delta+C\gamma_1\gamma,$$
$$\sigma C-\tau^{-1}c=\quad B+C\alpha-D\gamma^{-1},\quad \sigma D-\tau^{-1}d=C\gamma.$$

From the first four conditions,

$$b=-\gamma_1\tau A,\quad c=\gamma^{-1}\tau D,\quad a=\gamma_1^{-1}\tau B+\delta\tau A-\tau D,\quad d=\alpha\tau D-\gamma\tau C-\gamma_1\gamma\tau A.$$

In view of these, the last four conditions become

$$\gamma_1\gamma C=(\sigma-\delta)B,\quad D=-(\sigma-\delta)A,\quad B=(\sigma-\alpha)C,\quad \gamma_1\gamma A=-(\sigma-\alpha)D,$$

of which the last two follow from the first two in view of (58), which may be written $(\sigma-\alpha)(\sigma-\delta)=\gamma\gamma_1$. Expressing (34′) in terms of A and B, we get

$$\frac{1}{\tau}=[\gamma_1+\gamma^{-1}(\sigma-\delta)^2](A^2+\gamma_1^{-2}B^2)+[2\sigma-\delta+\gamma_1^{-1}\gamma^{-1}\alpha(\sigma-\delta)^2]AB$$
$$=\gamma^{-1}(\sigma-\delta)(2\sigma-\alpha-\delta)[A^2-\sigma(-\gamma_1^{-1}B)A+(-\gamma_1^{-1}B)^2].$$

Now $\sigma - \delta \neq 0$. If $2\sigma - \alpha - \delta \neq 0$, the condition merely determines τ whatever values, not both zero, in F be assigned to A and B. Indeed $A^2 - \sigma B_1 A + B_1^2$ is irreducible by the hypothesis. If $2\sigma - \alpha - \delta = 0$, (58) evidently has a double root. *Hence Q is conjugate with* (55) *if its characteristic determinant has in F a factor $\rho^2 - \sigma\rho + 1$, but not its square.*

19. Let next (57) be a perfect square, so that

$$(\alpha - \delta)^2 = -4\gamma\gamma_1. \tag{59}$$

According as F has not or has modulus 2, we have

$$\Delta(\rho) = \{\rho^2 - \tfrac{1}{2}(\alpha + \delta)\rho + 1\}^2, \quad \Delta(\rho) = \{\rho^2 + (\alpha^2 - \gamma\gamma_1)^{\frac{1}{2}} + 1\}^2. \tag{60}$$

Now whatever be the nature of (57), the only function, aside from a constant factor, which Q multiplies by κ, necessarily a root of the characteristic equation, is found to be

$$X_1 \equiv -\gamma\xi_1 + \kappa\gamma\gamma_1\eta_1 + (\kappa^2 - \kappa\delta + 1)\,\xi_2 + \kappa^{-1}\gamma(\kappa^2 - \kappa\delta + 1)\,\eta_2. \tag{61}$$

In the present special case, we introduce another function,

$$Y_1 \equiv \gamma\xi_1 + (\kappa^{-2} - 1)\,\xi_2 + \gamma(\delta - 2\kappa)\,\eta_2. \tag{62}$$

Denote by X_2, Y_2 the functions derived from X_1 and Y_1 respectively by replacing κ by κ^{-1}. In terms of these variables,* Q becomes, if (59) holds,

$$X_1' = \kappa X_1, \quad Y_1' = \kappa^{-1}Y_1 + \kappa^{-1}X_2, \quad X_2' = \kappa^{-1}X_2, \quad Y_2' = \kappa Y_2 + \kappa X_1. \tag{63}$$

Every transformation in F commutative with Q has the canonical form ($\bar{a}$ being the same function of κ^{-1} that a is of κ);

$$x_1 = aX_1, \quad y_1 = \bar{a}Y_1 + \bar{d}X_2, \quad x_2 = \bar{a}X_2, \quad y_2 = aY_2 + dX_1. \tag{64}$$

Then the transformation from $\xi_1, \eta_1, \xi_2, \eta_2$ to x_1, y_1, x_2, y_2 is the most general transformation T of variables transforming Q into the canonical form (63′), viz., (63) on the variables x_i, y_i. Consider the function

$$\Phi_{x,y} \equiv \begin{vmatrix} x_1 & y_1 \\ x_1' & y_1' \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_2' & y_2' \end{vmatrix},$$

* The determinant of transformation is found to equal $-\gamma^3\gamma_1(\kappa - \kappa^{-1})^4$. That it is not zero also follows from the Abelian character of T, as shown later.

where x_i', y_i' are the same linear functions of ξ_i', η_i', that x_i, y_i are of ξ_i, η_i. Then, as in §2, $\phi_{x,y}$ equals

$$C_{12}\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_1' & \eta_1' \end{vmatrix} + C_{13}\begin{vmatrix} \xi_1 & \xi_2 \\ \xi_1' & \xi_2' \end{vmatrix} + C_{14}\begin{vmatrix} \xi_1 & \eta_2 \\ \xi_1' & \eta_2' \end{vmatrix} + C_{23}\begin{vmatrix} \eta_1 & \xi_2 \\ \eta_1' & \xi_2' \end{vmatrix} + C_{24}\begin{vmatrix} \eta_1 & \eta_2 \\ \eta_1' & \eta_2' \end{vmatrix} + C_{34}\begin{vmatrix} \xi_2 & \eta_2 \\ \xi_2' & \eta_2' \end{vmatrix}.$$

For the transformation T, the value of C_{12} is

$$\begin{vmatrix} -a\gamma & a\varkappa\gamma\gamma_1 \\ \bar{a}\gamma - \bar{d}\gamma & \bar{d}\varkappa^{-1}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} -\bar{a}\gamma & \bar{a}\varkappa^{-1}\gamma\gamma_1 \\ a\gamma - d\gamma & d\varkappa\gamma\gamma_1 \end{vmatrix} = -\gamma^2\gamma_1\{A + a\bar{a}(\varkappa + \varkappa^{-1})\},$$

where $$A = (\bar{a}d - a\bar{d})(\varkappa - \varkappa^{-1}).$$

Similarly, we find that

$$C_{13} = -\gamma\{A(\varkappa + \varkappa^{-1} - \delta) + a\bar{a}(2\varkappa^2 + 2\varkappa^{-2} - \delta(\varkappa + \varkappa^{-1})\},$$
$$C_{14} = 0, \quad C_{23} = 0, \quad C_{24} = \gamma\gamma_1 C_{13},$$
$$C_{34} = -\gamma A(\varkappa + \varkappa^{-1} - \delta)^2 + \gamma(\varkappa + \varkappa^{-1} - \delta)a\bar{a}(2 + \delta\varkappa + \delta\varkappa^{-1} - 3\varkappa^2 - 3\varkappa^{-2}).$$

The discriminant $\tau \equiv (\varkappa - \varkappa^{-1})^2$ is not zero by hypothesis. Set

$$d/a = \lambda + \varkappa\mu, \text{ whence } \bar{d}/\bar{a} = \lambda + \varkappa^{-1}\mu.$$

Hence

$$A = a\bar{a}\left(\frac{d}{a} - \frac{\bar{d}}{\bar{a}}\right)(\varkappa - \varkappa^{-1}) = a\bar{a}\tau\mu, \quad C_{12} = -\gamma^2\gamma_1 a\bar{a}(\tau\mu + \varkappa + \varkappa^{-1}),$$

$$C_{13} = -\gamma a\bar{a}\{\tau\mu(\varkappa + \varkappa^{-1} - \delta) + 2\tau + 4 - \delta(\varkappa + \varkappa^{-1})\},$$
$$C_{34} = -\gamma(\varkappa + \varkappa^{-1} - \delta)a\bar{a}\{\tau\mu(\varkappa + \varkappa^{-1} - \delta) + 3\tau + 4 - \delta(\varkappa + \varkappa^{-1})\}.$$

From (57), we may write the characteristic equation

$$\gamma\gamma_1 = (\rho + \rho^{-1} - \alpha)(\rho + \rho^{-1} - \delta). \qquad (65)$$

Hence C_{12} may be written

$$C_{12} = -\gamma a\bar{a}(\varkappa + \varkappa^{-1} - \alpha)\{\tau\mu(\varkappa + \varkappa^{-1} - \delta) + \tau + 4 - \delta(\varkappa + \varkappa^{-1})\},$$

By (65), $\varkappa + \varkappa^{-1} - \delta \neq 0$. Hence we can determine μ in the field to make $C_{13} = 0$. It then follows immediately that

$$C_{12} = \gamma a\bar{a}(\varkappa + \varkappa^{-1} - \alpha)\tau, \quad C_{34} = -\gamma a\bar{a}(\varkappa + \varkappa^{-1} - \delta)\tau.$$

When F has modulus 2, $\alpha = \delta$ and $C_{12} = C_{34}$. If F does not have modulus 2,

$\varkappa + \varkappa^{-1} \equiv \frac{1}{2}(\alpha + \delta)$ by $(60)_1$, whence $C_{13} = C_{34}$. Hence, for any field F, it follows that

$$\phi_{x,y} = C\phi_{\xi,\eta}, \quad C \equiv -\gamma a\bar{a}(\varkappa + \varkappa^{-1} - \delta)\tau. \tag{66}$$

Hence T is a general Abelian transformation (§2) in the enlarged field $F(\varkappa)$. Let Q_1 be a second transformation (56) with the same characteristic determinant (60) as the given Q. It follows from the theory of canonical forms with conjugate variables that Q can be transformed into Q_1 by a general Abelian transformation in F. The latter will be a special Abelian transformation in F if and only if $C_1 = C$. This condition can be satisfied by choice of a and a_1 if the ratio of the two values of $\gamma(\varkappa + \varkappa^{-1} - \delta)$ is of the form $e\bar{e}$, where e belongs to $F(\varkappa)$. If F be the $GF[p^n]$, $e\bar{e} = e^{p^n+1}$ can be made to assume any given value in the $GF[p^n]$. If F be the field of all real numbers, $e\bar{e}$ can evidently be made to take any positive value in F, but not a negative value.

THEOREM.—*If two transformations Q have as common characteristic determinant the square of a factor irreducible in F and if F is the $GF[p^n]$, they are conjugate within $SA[4, p^n]$. If F is the field of all real numbers, there are two distinct sets of conjugates.**

Corollary. If F is the $GF[p^n]$, C can be made equal to unity by choice of a, so that Q can be reduced to its canonical form (63) by a special Abelian transformation in the $GF[p^{2n}]$.

20. We now consider Q in the two remaining cases:

(i) $\Delta(\rho)$ irreducible in the field F;

(ii) $\Delta(\rho)$ the product to two irreducible factors $\rho^2 - e\rho + f$, $f \neq 1$.

In either case, the roots of $\Delta(\rho) = 0$ are all distinct and may be designated $\varkappa$, $\varkappa^{-1}$, λ, λ^{-1}. Hence by (57),

$$\varkappa + \varkappa^{-1} + \lambda + \lambda^{-1} = \alpha + \delta. \tag{67}$$

In case (ii), $\varkappa$ and λ are the roots of a quadratic equation irreducible in F. Consider X_1 defined by (61). Denote by Y_1, X_2, Y_2, the functions derived from X_1 upon replacing $\varkappa$ by $\varkappa^{-1}$, λ, λ^{-1}, respectively. In case (i), X_1, Y_1, X_2, Y_2 are conjugate with respect to F. In case (ii) X_1 and X_2 are conjugate with respect

* For other infinite fields, the question of conjugacy can be determined by introducing the R_{ij} as in the next section.

to F; likewise, Y_1 and Y_2. In each case the four functions satisfy the requirements as to conjugacy in the general theory of canonical forms,* and Q takes the canonical form

$$X_1' = \varkappa X_1, \quad Y_1' = \varkappa^{-1} Y_1, \quad X_2' = \lambda X_2, \quad Y_2' = \lambda^{-1} Y_2. \tag{68}$$

We proceed to prove that the transformation of variables from the ξ_i, η_i to X_i, Y_i satisfies the conditions

$$R_{13} = R_{14} = R_{23} = R_{24} = 0,$$

where R_{ij} is defined in §2. We have

$$R_{13} = \begin{vmatrix} -\gamma & \varkappa\gamma\gamma_1 \\ -\gamma & \lambda\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \varkappa^2 - \varkappa\delta + 1 & \varkappa^{-1}\gamma(\varkappa^2 - \varkappa\delta + 1) \\ \lambda^2 - \lambda\delta + 1 & \lambda^{-1}\gamma(\lambda^2 - \lambda\delta + 1) \end{vmatrix}$$
$$= \gamma(\varkappa - \lambda)\{\gamma\gamma_1 + (\varkappa + \varkappa^{-1} - \delta)(\lambda + \lambda^{-1} - \delta)\}.$$

Eliminating $\gamma\gamma_1$ by means of (65) for $\rho = \varkappa$, we get

$$R_{13} = \gamma(\varkappa - \lambda)(\varkappa + \varkappa^{-1} - \delta)(\varkappa + \varkappa^{-1} + \lambda + \lambda^{-1} - \alpha - \delta).$$

Hence $R_{13} = 0$ by (67). Replacing λ by λ^{-1}, we get $R_{14} = 0$. Replacing $\varkappa$ by $\varkappa^{-1}$, we get $R_{23} = 0$. Making both replacements, we get $R_{24} = 0$. Replacing λ by $\varkappa^{-1}$, we get

$$R_{12} = \gamma(\varkappa - \varkappa^{-1})\, C, \quad C \equiv (\varkappa + \varkappa^{-1} - \delta)(2\varkappa + 2\varkappa^{-1} - \alpha - \delta). \tag{69}$$

Replacing $\varkappa$ by λ in the preceding, we get

$$R_{34} = \gamma(\lambda - \lambda^{-1})\, C_1, \quad C_1 \equiv (\lambda + \lambda^{-1} - \delta)(2\lambda + 2\lambda^{-1} - \alpha - \delta). \tag{70}$$

In either case (i) or case (ii), we have $C \neq 0$, $C_1 \neq 0$.

For case (ii), we introduce in place of X_1, X_2 the variables

$$x_1 = X_1 \div \gamma(\varkappa - \varkappa^{-1})\, C, \quad x_2 = X_2 \div \gamma(\lambda - \lambda^{-1})\, C_1.$$

Then x_1 and x_2 are conjugate with respect to F, and (68) becomes

$$x_1' = \varkappa x_1, \quad Y_1' = \varkappa^{-1} Y_1, \quad x_2' = \lambda x_2, \quad Y_2' = \lambda^{-1} Y_2. \tag{68'}$$

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* American Journal, Vol. XXIV, pp. 101-108.

For the transformation of variables from ξ_i, η_i to x_i, Y_i, we have

$$R'_{13} = R'_{14} = R'_{23} = R'_{24} = 0, \quad R'_{12} = 1, \quad R'_{34} = 1,$$

so that it is a special Abelian transformation in the extended field $F(\kappa, \lambda)$.

THEOREM.—*Within the group $SA(4, F)$ two transformations are conjugate if they have the same characteristic determinant which is the product of two irreducible factors $\rho^2 - a\rho + b$, $b \neq 1$.*

For case (i), we introduce in place of X_1, Y_1, X_2, Y_2 the variables*

$$x_1 = X_1/f(\kappa), \quad y_1 = Y_1/f(\kappa^{-1}), \quad x_2 = X_2/f(\lambda), \quad y_2 = Y_2/(\lambda^{-1}),$$

where $f(\rho) = a\rho^3 + b\rho^2 + c\rho + d$, a, b, c, d being undetermined elements of F not all zero. Then x_1, y_1, x_2, y_2 are conjugate with respect to F and (68) becomes

$$x'_1 = \kappa x_1, \quad y'_1 = \kappa^{-1} y_1, \quad x'_2 = \lambda x_2, \quad y'_2 = \lambda^{-1} y_2. \tag{68''}$$

Let T denote the transformation of variables from ξ_i, η_i to x_i, y_i. For it,

$$R''_{13} = R''_{14} = R''_{23} = R''_{24} = 0, \quad R''_{12} = \frac{\gamma(\kappa - \kappa^{-1})\, C}{f(\kappa) f(\kappa^{-1})}, \quad R''_{34} = \frac{\gamma(\lambda - \lambda^{-1})\, C_1}{f(\lambda) f(\lambda^{-1})}.$$

For brevity, set $R''_{12} = \phi(\kappa)$. Then $R''_{24} = \phi(\lambda)$. By the Corollary to §3,

$$\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_1^* & \eta_1^* \end{vmatrix} + \begin{vmatrix} \xi_2 & \eta_2 \\ \xi_2^* & \eta_2^* \end{vmatrix} = \frac{1}{\phi(\kappa)} \begin{vmatrix} x_1 & y_1 \\ x_1^* & y_1^* \end{vmatrix} + \frac{1}{\phi(\lambda)} \begin{vmatrix} x_2 & y_2 \\ x_2^* & y_2^* \end{vmatrix},$$

where x_i^*, y_i^* are the same functions of ξ_i^*, η_i^* that x_i, y_i are of ξ_i, η_i, so that T affects the two sets of variables cogrediently. It follows that two transformations Q, with the same irreducible characteristic determinant $\Delta(\rho)$, are conjugate within $SA(4, F)$, if (and, at least for normal fields,* only if) it be possible to choose $f(\rho)$ so that $\phi(\kappa)$ shall depend only upon the invariant $\Delta(\rho)$ and not upon the individual coefficients of Q. Now

$$\phi(\kappa) = (\kappa - \kappa^{-1})(2\kappa + 2\kappa^{-1} - \alpha - \delta) D, \quad D \equiv \gamma(\kappa + \kappa^{-1} - \delta)/f(\kappa) f(\kappa^{-1}).$$

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* The field $F(\kappa, \lambda)$ may be a normal field, i. e., $\lambda =$ rat. func. (κ), with coefficients in F. This is the case when F is the $GF[p^n]$. It would be interesting to determine, for $F(\kappa, \lambda)$ not normal, whether or not it is possible to retain the advantage of conjugate variables and yet employ $f(\kappa, \lambda)$ in place of $f(\kappa)$. By (67), $\lambda^2 - \lambda(\alpha + \delta - \kappa - \kappa^{-1}) + 1 = 0$, so that $f(\kappa, \lambda)$ may be written $f_1(\kappa) + \lambda f_2(\kappa)$.

When F is the $GF[p^n]$, x^{-1} must equal x^{p^n}, $x^{p^{2n}}$ or $x^{p^{3n}}$, since these three together with x furnish the distinct roots of $\Delta(\rho)=0$. But $x^{-1}=x^{p^n}$ would require that $x^{p^{2n}}=x^{-p^n}=x$. Again $x^{-1}=x^{p^{3n}}$ would require that $x^{-p^n}=x^{p^{4n}}=x$. Hence must $x^{-1}=x^{p^{2n}}$, so that

$$f(x^{-1})=[f(x)]^{p^{2n}},\quad K\equiv x+x^{-1}=\text{element of } GF[p^{2n}].$$

To make $D=1$, it suffices to take as $f(x)$ a root of

$$f^{p^{2n}+1}=\gamma(K-\delta),$$

all of whose roots f belong to the $GF[p^{4n}]$. Indeed, the power $p^{2n}-1$ of the second member is unity, not being zero in view of the irreducibility of $\Delta(\rho)$. *Hence two transformations Q having the same irreducible $\Delta(\rho)$ are conjugate within $SA(4, p^n)$.*

For a general field F, we have, by (65),

$$K^2=\epsilon K+\tau,\quad \epsilon\equiv\alpha+\delta,\quad \tau=\gamma\gamma_1-\alpha\delta\neq 0,\quad K=x+x^{-1}.$$

Then $f(x)f(x^{-1})$ takes the form $rK+s$, where

$$\begin{cases} r=ab+bc+cd+(ac+bd)\,\epsilon+ad\,(\epsilon^2+\tau-3), \\ s=a^2+b^2+c^2+d^2+(ac+bd)(\tau-2)+ad\epsilon\tau. \end{cases}\tag{71}$$

Hence

$$\frac{1}{D}=\frac{rK+s}{\gamma(K-\delta)}=\frac{(rK+s)(K-\alpha)}{\gamma^2\gamma_1}=\frac{K(s+r\delta)+r\tau-s\alpha}{\gamma^2\gamma_1}.$$

Then $D=1$ if and only if $r=\gamma$, $s=-\delta\gamma$. Hence any two Q with the same $\Delta(\rho)$ are conjugate if equations (71) are solvable in F for a, b, c, d, whatever values r and s have in F. They are equivalent to

$$P\,\{ac+bd+(\epsilon+2)\,ad\}+(a+b+c+d)^2=s+2r,$$
$$N\,\{ac+bd+(\epsilon-2)\,ad\}+(a-b+c-d)^2=s-2r,$$

where

$$P=\tau-4+2\epsilon=\gamma\gamma_1-(2-\alpha)(2-\delta),$$
$$N=\tau-4-2\epsilon=\gamma\gamma_1-(-2-\alpha)(-2-\delta)$$

are not zero in view of the irreducibility of $\gamma\gamma_1-(K-\alpha)(K-\delta)$.

We may attack the problem otherwise. If Q and Q' have the same $\Delta(\rho)$ they are conjugate if $D=D'$, viz., if

$$s'+r'\delta'=\mu(s+r\delta),\quad r'\tau-s'\alpha'=\mu(r\tau-s\alpha),\quad \mu=\gamma'^2\gamma_1'/\gamma^2\gamma_1,$$

for suitable values of $a, b, c, d, a', b', c', d'$ in F. We thus have two quadratic conditions on 8 unknown quantities.

21. We next consider transformation S_1 of §17 for $m=3$. Applying certain Abelian conditions, we have

$$S_1=\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & \gamma\gamma_{22} & \alpha_{22} & \gamma_{23} & \alpha_{23} & \gamma_{23} \\ 0 & \gamma\delta_{22} & \beta_{22} & \delta_{22} & \beta_{23} & \delta_{23} \\ 0 & \gamma\gamma_{32} & \alpha_{32} & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ 0 & \gamma\delta_{32} & \beta_{32} & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}. \tag{72}$$

If $\gamma_{32}=\delta_{32}=0$, we may transform S_1 by a transformation on ξ_3 and η_3 and obtain a like transformation with also $\gamma_{23}=0$. If γ_{32} and δ_{32} are not both zero, we may transform S_1 by a transformation on ξ_3 and η_3 and make $\gamma_{32}=0$, $\delta_{32}\neq 0$. We therefore treat S_1 under two cases:

$$\text{(I).}\quad \gamma_{32}=\delta_{32}=\gamma_{23}=0;\qquad \text{(II).}\quad \gamma_{32}=0,\quad \delta_{32}\neq 0.$$

Case I. If also $\gamma_{22}=0$, the Abelian conditions C_{34} and C_{45} give $\alpha_{22}\delta_{22}=1$, $\delta_{22}\alpha_{23}=0$, whence $\xi_2'=\alpha_{22}\xi_2$, so that $\Delta(\rho)=0$ has a root α_{22} in F. Hence $\gamma_{22}\neq 0$. Then Abelian condition C_{46} gives $\delta_{23}=0$. Transforming S_1 by $\eta_2'=\eta_2+\lambda\xi_2$, $\delta_{22}+\lambda\gamma_{22}=0$, we obtain a transformation S_1' of the form S_1 with $\gamma_{32}=\delta_{32}=\gamma_{23}=0$, $\delta_{23}=0$, $\gamma_{22}\neq 0$, and also $\delta_{22}=0$. Then Abelian conditions C_{45} and R_{35} give $\beta_{23}=0$, $\alpha_{23}\gamma_{33}-\gamma_{22}\alpha_{32}=0$. If $\gamma_{33}=0$, then $\alpha_{32}=0$, $\xi_3'=\alpha_{33}\xi_3$, and $\Delta(\rho)=0$ has a root in F. Hence $\gamma_{33}\neq 0$. Transforming S_1' by $\eta_3'=\eta_3+\lambda\xi_3$, $\delta_{33}+\lambda\gamma_{33}=0$, we obtain a like transformation with also $\delta_{33}=0$. Then $\beta_{32}=0$ by Abelian condition R_{36}. The resulting Abelian transformation is

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & \gamma\gamma_{22} & \alpha_{22} & \gamma_{23} & \alpha_{23} & 0 \\ 0 & 0 & -\gamma_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22}^{-1}\alpha_{23}\gamma_{33} & 0 & \alpha_{33} & \gamma_{33} \\ 0 & 0 & 0 & 0 & -\gamma_{33}^{-1} & 0 \end{pmatrix}.$$

If $\alpha_{23}=0$, its transform by M_3P_{13} is of the form (55). If $\alpha_{23}\neq 0$, its transform by $T_{3,\alpha_{23}}$ is of like form with $\alpha_{23}=1$, viz., (92) below.

Case II. If $\gamma_{22}\neq 0$, the transform of S_1 by $R_{2,3,\lambda}$, $\delta_{32}-\lambda\gamma_{22}=0$, is of the form S_1 with $\gamma_{32}=\delta_{32}=0$, and hence falls under case I. Let next $\gamma_{22}=0$. Transforming by $T_{3,\delta_{23}}$ we obtain an S_1 with $\gamma_{32}=\gamma_{22}=0$, $\delta_{32}=1$. Transforming the latter by $Q_{3,2,\delta_{22}}$, we obtain a like transformation S' with also $\delta_{22}=0$. Then Abelian conditions C_{34}, C_{45}, C_{46} give $\alpha_{32}=1$, $\alpha_{33}=0$, $\gamma_{33}=0$.

If S' has $\gamma_{23}\neq 0$, its transform by

$$\eta_2'=\eta_2+\lambda\xi_2,\quad \eta_3'=\eta_3+\mu\xi_3,\quad \alpha_{23}-\mu\gamma_{23}=0,\quad \delta_{23}+\lambda\gamma_{23}=0,$$

is of the form S' with $\alpha_{23}=\delta_{23}=0$. Then $R_{2,3,\beta_{22}}$, transforms the latter into

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & g \\ 0 & 0 & 0 & 0 & -g^{-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 1 & g^{-1}\alpha & 0 \end{pmatrix} \tag{73}$$

which is studied in §§24–26. Its characteristic determinant is

$$\rho^6-(\rho^5+\rho)(\delta+\alpha)+(\rho^4+\rho^2)(1+\delta\alpha-\beta g)-\rho^3(2\alpha+\gamma g-\delta\beta g)+1. \tag{74}$$

Let next $\gamma_{23}=0$ in S'. Then $\alpha_{23}\delta_{23}=1$, $\alpha_{22}+\alpha_{23}\delta_{33}=0$ by Abelian conditions. If $\alpha_{22}\neq 0$, we transform by $\eta_2'=\eta_2+\lambda\xi_2$, $\eta_3'=\eta_3+\mu\xi_3$, where $\beta_{22}+\lambda\alpha_{22}=0$, $\beta_{23}+\lambda\alpha_{23}-\mu\delta_{23}=0$, and obtain a transformation which in view of the Abelian conditions may be written

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & g^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & g \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 1 & 0 & -g\alpha \end{pmatrix}. \tag{75}$$

It is studied in §23. Its characteristic determinant is

$$\rho^6-(\rho^5+\rho)(\delta+\alpha-g\alpha)+(\rho^4+\rho^2)(1+\delta\alpha-g\delta\alpha-g-g^{-1}-g\alpha^2)$$
$$-\rho^3(2\alpha-2g\alpha-\delta g-\delta g^{-1}-g\delta\alpha^2)+1. \tag{76}$$

If $\alpha_{22}=0$, then $\delta_{33}=0$, $\beta_{22}=\delta_{23}\beta_{33}$. Transforming by $\eta_3'=\eta_3-\beta_{32}\xi_3$, we obtain

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & 0 & \beta' & \alpha^{-1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 1 & \alpha\beta & 0 \end{pmatrix}. \tag{77}$$

Its characteristic determinant is seen to equal

$$(\rho^2-\rho\delta+1)(\rho^2-\alpha)(\rho^2-\alpha^{-1}). \tag{78}$$

22. The last result suggests that we determine the conditions under which (77) is conjugate within G with a transformation (55) which replaces ξ_1 by $\tau\eta_1$ and η_1 by $-\tau^{-1}\xi_1+\delta\eta_1$. The conditions that (77) shall replace X and Y, defined by (33), by respectively τY and $-\tau^{-1}X+\delta Y$, are seen to be

$$\tau A=-b\gamma^{-1},\quad \tau B=a\gamma+b\delta+j\gamma,\quad \tau C=b+e+d\beta,$$
$$\tau D=j,\quad \tau E=c\alpha+d\beta'+j\alpha\beta,\quad \tau J=d\alpha^{-1},$$
$$\delta A-\tau^{-1}a=-B\gamma^{-1},\quad \delta B-\tau^{-1}b=A\gamma+B\delta+J\gamma,$$
$$\delta C-\tau^{-1}c=B+E+D\beta,$$
$$\delta D-\tau^{-1}d=J,\quad \delta E-\tau^{-1}e=C\alpha+D\beta'+J\alpha\beta\quad \delta J-\tau^{-1}j=D\alpha^{-1}.$$

From the first set of six conditions, we get

$$b=-\gamma\tau A,\quad j=\tau D,\quad d=\alpha\tau J,\quad e=\tau C+\gamma\tau A-\alpha\beta\tau J,$$
$$a=\gamma^{-1}\tau B+\delta\tau A-\tau D,\quad c=\alpha^{-1}\tau E-\beta'\tau J-\beta\tau D.$$

Substituting these values in the second set of six, we get

$$D=0,\quad J=0,\quad B=\delta C-E-\alpha^{-1}E,\quad \gamma A=\delta E-C-\alpha C.$$

Hence $d=j=0$. Abelian condition (34) thus becomes

$$aB-bA=1,\quad \text{or}\quad \gamma^{-1}\tau(B^2+\gamma\delta AB+\gamma^2A^2)=1. \tag{79}$$

Now C and E can be chosen in the field to make $B\neq 0$, unless $\delta=0$, $\alpha=-1$ (when also $A\equiv 0$). For $B\neq 0$, the condition may be written

$$\gamma^{-1}\tau B^2[1-\delta(-\gamma AB^{-1})+(-\gamma AB^{-1})^2]=1. \tag{79'}$$

This merely serves to determine τ in F for any given values of A and B ($B\neq 0$)

in F, since $1-\delta\rho+\rho^2$ is irreducible in F by hypothesis. *Hence* (77) *is conjugate within G with a transformation* (55) *if, and only if, δ and $\alpha+1$ are not both zero.*

It remains to consider (77) in the case $\delta=0$, $\alpha=-1$, whence (78) becomes $(\rho^2+1)^3$. Since ρ^2+1 shall be irreducible, F cannot have modulus 2. Then $R_{2,3,-\beta/2}$ transforms (77) into a similar transformation with $\beta=\delta=0$, $\alpha=-1$. Now $L'_{3,-\beta'}$ transforms the latter into (75) for $\delta=\alpha=0$, $g=-1$, $\beta=-\beta'$.

23. For type (75), the characteristic determinant (76) factors into

$$(\rho^2-\rho\delta+1)(\rho^2-\alpha\rho-g^{-1})(\rho^2+g\alpha\rho-g). \tag{76'}$$

The conditions that (75) shall replace X and Y, defined by (33), by respectively τY and $-\tau^{-1}X+\delta Y$ are seen (as in §22) to reduce to

$$\begin{gathered}
b=-\gamma\tau A,\quad j=\tau D,\quad c=g\tau E,\quad a=\gamma^{-1}\tau B+\delta\tau A-\tau D,\\
d=g^{-1}\tau J+\alpha\tau D,\quad e=\tau C+\gamma\tau A-g\alpha\tau E-\beta\tau D;\\
D=0,\quad J=0,\quad B=C(\delta-\alpha)-E(g+1),\quad \gamma A=-C(g^{-1}+1)+E(\delta+g\alpha).
\end{gathered}$$

The Abelian condition (34) reduces to (79). Now C and E can be chosen in F to make $B\neq 0$ unless $\delta=\alpha$, $g=-1$ (when also $A\equiv 0$). *Hence* (75) *is conjugate within G with a transformation* (55) *if, and only if, $\delta-\alpha$ and $g+1$ are not both zero.*

24. Now $\rho^2-\sigma\rho+1$ is a factor of (74) if, and only if,

$$\chi(\sigma)\equiv\sigma^3-\sigma^2(\alpha+\delta)+\sigma(\alpha\delta-\beta g-2)+2\delta-\gamma g+\beta\delta g=0. \tag{80}$$

Supposing this to have a root σ in the field F, and that $\rho^2-\sigma\rho+1$ is irreducible, we determine the conditions under which (73) is conjugate with a transformation (55). Let X and Y be defined by (33). The conditions that (73) shall replace X by τY and Y by $-\tau^{-1}X+\sigma Y$ are

$$\begin{gathered}
\tau A=-b\gamma^{-1},\quad \tau B=a\gamma+b\delta+j\gamma,\quad \tau C=b+ca+e+j\beta,\quad \tau D=j,\\
\tau E=-g^{-1}d+g^{-1}aj,\quad \tau J=cg,\quad \sigma A-\sigma^{-1}a=-B\gamma^{-1},\\
\sigma B-\tau^{-1}b=A\gamma+B\delta+J\gamma,\quad \sigma C-\tau^{-1}c=B+C\alpha+E+J\beta,\\
\sigma D-\tau^{-1}d=J,\quad \sigma E-\tau^{-1}e=-g^{-1}D+g^{-1}\alpha J,\quad \sigma J-\tau^{-1}j=Cg.
\end{gathered}$$

From the first six conditions,

$$\left.\begin{array}{l} b=-\gamma\tau A\,,\quad j=\tau D\,,\quad c=g^{-1}\tau J\,,\quad a=\gamma^{-1}\tau B+\delta\tau A-\tau D\,,\\ d=-g\tau E+\alpha\tau D,\quad e=\tau C+\gamma\tau A-g^{-1}\alpha\tau J-\beta\tau D. \end{array}\right\} \tag{81}$$

Substituting these values in the final six conditions, we get

$$\left.\begin{array}{l} D=-(\sigma-\delta)\,A,\quad J=\gamma^{-1}(\sigma-\delta)\,B,\quad C=g^{-1}(\sigma-\delta)(\gamma^{-1}\sigma B+A),\\ \qquad\qquad\qquad\qquad\qquad\qquad E=g^{-1}\,(\sigma-\delta)(\gamma^{-1}B+\sigma A-\alpha A),\\ E=g^{-1}(\sigma-\alpha)(\sigma-\delta)(\gamma^{-1}\sigma B+A)-(g^{-1}+\beta)(\sigma-\delta)\,\gamma^{-1}B-B,\\ \sigma E=g^{-1}(\sigma-\delta)(\gamma^{-1}\sigma B+A)+(g^{-1}+\beta)(\sigma-\delta)\,A+\gamma A. \end{array}\right\} \tag{82}$$

The difference of the first two expressions for E vanishes in view of (80). Multiplying the second expression for E by σ and subtracting the given expression for σE, we obtain $\gamma^{-1}\sigma B+A$ times (80). Hence the three values for E are consistent. In view of (82), (81) become

$$\left.\begin{array}{l} b=-\gamma\tau A,\quad c=g^{-1}\gamma^{-1}(\sigma-\delta)\,\tau B,\quad j=-(\sigma-\delta)\,\tau A,\\ a=\gamma^{-1}\tau B+\sigma\tau A,\quad d=-\gamma^{-1}(\sigma-\delta)\,\tau B-\sigma\,(\sigma-\delta)\,\tau A,\\ e=[\gamma+(\beta+g^{-1})(\sigma-\delta)]\,\tau A+g^{-1}\gamma^{-1}(\sigma-\alpha)(\sigma-\delta)\,\tau B. \end{array}\right\} \tag{83}$$

Then (34) becomes,* on substitution of the values from (82) and (83),

$$\frac{1}{\tau}=M(A^2+\sigma AB\gamma^{-1}+B^2\gamma^{-2})\,,\quad M\equiv\gamma+(2g^{-1}\sigma-g^{-1}\alpha)(\sigma-\delta)^2.$$

If $M\neq 0$, this equation determines τ, whatever be the values of A and B, not both zero (see §22). Eliminating γ by means of (80),

$$M=g^{-1}(\sigma-\delta)(3\sigma^2-2\sigma\alpha-2\sigma\delta+\alpha\delta-2-\beta g)\,.$$

The final factor is the derivative of the left member of (80). Also, $\sigma\neq\delta$ in view of (80). Hence $M=0$ if, and only if, σ is a double root of (80). In this case, the third root of (80) may be determined by rational process, and hence belongs to F. If it differs from σ, it is a simple root and hence leads to an $M\neq 0$.

THEOREM.—*If* (74) *has in the field an irreducible factor* $\rho^2-\sigma\rho+1$ *and differs from* $(\rho^2-\sigma\rho+1)^3$, *then* (73) *is conjugate within* G *with a transformation* (55).

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* The coefficient of $\gamma^{-1}AB$ is initially $\gamma\,(2\sigma-\delta)+(\beta+2g^{-1}+g^{-1}\sigma^2)(\sigma-\delta)^2$. Replacing $\beta g\,(\sigma-\delta)$ by its value $\sigma^3-\sigma^2\,(\alpha+\delta)+\sigma\,(\alpha\delta-2)+2\delta-\gamma g$ from (80), we get σM.

25. Suppose that the characteristic determinant of a special Abelian transformation has in the field F neither a linear factor nor a quadratic factor of the form $\rho^2-\sigma\rho+1$. It therefore has no quadratic factor whatever. For if $\rho^2-e\rho+f$, $f\neq 1$, be a factor then, by §5, $\rho^2-\frac{e}{f}\rho+\frac{1}{f}$ would be a factor, so that there would be a third factor of the form $\rho^2-\sigma\rho+1$, with σ in F. If an irreducible cubic factor vanishes for $\varkappa$, λ, μ, there occurs a second irreducible cubic factor vanishing for $\varkappa^{-1}$, λ^{-1}, μ^{-1} and the six quantities

$$\varkappa,\ \lambda,\ \mu,\quad \varkappa^{-1},\ \lambda^{-1},\ \mu^{-1} \tag{84}$$

are all distinct. Thus, if $\varkappa^{-1}=\lambda$, then μ would belong to F. Finally, if $\Delta(\rho)$ be irreducible, it vanishes for 6 distinct values (84).

26. In view of §§24 and 25, it remains to discuss (73) when (74) vanishes for 6 distinct values (84). Now (73) replaces

$$X_1\equiv a\xi_1+b\eta_1+c\xi_2+d\eta_2+e\xi_3+j\eta_3$$

by $\varkappa X_1$ if, and only if,

$$\varkappa a=-b\gamma^{-1},\quad \varkappa b=a\gamma+b\delta+j\gamma,\quad \varkappa c=b+c\alpha+e+\beta j,$$
$$\varkappa d=j,\quad \varkappa e=-g^{-1}d+g^{-1}\alpha j,\quad \varkappa j=cg.$$

Now $a\neq 0$, since $a=0$ requires that $X_1\equiv 0$. For $a=1$, we have

$$X_1=\xi_1-\gamma\varkappa\eta_1+g^{-1}\varkappa D\xi_2+\varkappa^{-1}D\eta_2+g^{-1}\varkappa^{-1}(\alpha-\varkappa^{-1})D\xi_3+D\eta_3, \tag{85}$$

where $D\equiv-\varkappa^2+\varkappa\delta-1$, and $\varkappa$ is a root of (74). Let Y_1, X_2, Y_2, X_3, Y_3 be the functions derived from (85) upon replacing $\varkappa$ by $\varkappa^{-1}$, λ, λ^{-1}, μ, μ^{-1}, respectively. Let $D'\equiv-\lambda^2+\lambda\delta-1$. Then

$$R_{13}=\begin{vmatrix}1 & -\gamma\varkappa\\ 1 & -\gamma\lambda\end{vmatrix}+\begin{vmatrix}g^{-1}\varkappa D & \varkappa^{-1}D\\ g^{-1}\lambda D' & \lambda^{-1}D'\end{vmatrix}+\begin{vmatrix}g^{-1}\varkappa^{-1}(\alpha-\varkappa^{-1})D & D\\ g^{-1}\lambda^{-1}(\alpha-\lambda^{-1})D' & D'\end{vmatrix}$$
$$=(\varkappa-\lambda)\{\gamma+g^{-1}(\delta-\varkappa-\varkappa^{-1})(\delta-\lambda-\lambda^{-1})(\varkappa+\lambda)$$
$$+g^{-1}(\delta-\varkappa-\varkappa^{-1})(\delta-\lambda-\lambda^{-1})(\varkappa^{-1}+\lambda^{-1}-\alpha)\},$$

$$R_{13}=g^{-1}(\varkappa-\lambda)\{g\gamma+(\delta-\varkappa-\varkappa^{-1})(\delta-\lambda-\lambda^{-1})(\varkappa+\varkappa^{-1}+\lambda+\lambda^{-1}-\alpha)\}. \tag{86}$$

In view of the notation (84) for the roots of (74), we have

$$\varkappa+\varkappa^{-1}+\lambda+\lambda^{-1}+\mu+\mu^{-1}=\alpha+\delta. \tag{87}$$

Hence

$$R_{13} = g^{-1}(\kappa - \lambda)\{g\gamma + (\delta - \kappa - \kappa^{-1})(\delta - \lambda - \lambda^{-1})(\delta - \mu - \mu^{-1})\}.$$

Now $\kappa + \kappa^{-1}$, $\lambda + \lambda^{-1}$, $\mu + \mu^{-1}$ are the roots of (80). Hence

$$(\delta - \kappa - \kappa^{-1})(\delta - \lambda - \lambda^{-1})(\delta - \mu - \mu^{-1}) = \chi(\delta), \quad \gamma g + \chi(\delta) \equiv 0. \quad R_{13} = 0.$$

Replacing κ, λ by any two values (84), not inverse to each other,

$$R_{ij} = 0, \tag{88}$$

$$[i, j = 1, \ldots, 6, \text{ except for } (i, j) = (1, 2), (3, 4), (5, 6)].$$

Replacing λ by κ^{-1} in (86), we get*

$$R_{12} = g^{-1}(\kappa - \kappa^{-1})\{g\gamma + (\delta - \kappa - \kappa^{-1})^2(2\kappa + 2\kappa^{-1} - \alpha)\}. \tag{89}$$

Let $R_{12} = r(\kappa)$. Then $R_{34} = r(\lambda)$, $R_{56} = r(\mu)$.

Let first $\Delta(\rho)$ be the product of two irreducible cubic factors in F, the first with the roots κ, λ, μ, the second with the roots κ^{-1}, λ^{-1}, μ^{-1} (§25). Let

$$x_1 = X_1/r(\kappa), \quad x_2 = X_2/r(\lambda), \quad x_3 = X_3/r(\mu),$$

so that x_1, x_2, x_3 are conjugate with respect to F; likewise Y_1, Y_2 Y_3. Then (73) takes the canonical form

$$x_1' = \kappa x_1, \quad Y_1' = \kappa^{-1} Y_1, \quad x_2 = \lambda x_2, \quad Y_2' = \lambda^{-1} Y_2, \quad x_3' = \mu x_3, \quad Y_3 = \mu^{-1} Y_3. \tag{90}$$

For the transformation of variables from ξ_i, η_i to x_i, Y_i, we have

$$R_{12}' = R_{34}' = R_{56}' = 1, \quad R_{ij}' = 0 \text{ [except for } (i, j) = (1, 2), (3, 4), (5, 6)],$$

so that it is a special Abelian transformation in the field $F(\kappa, \lambda, \mu)$.

THEOREM.—*Within the group* $SA(6, F)$, *two transformations* (73) *are conjugate if they have as common characteristic determinant the product of two irreducible cubic factors.*

Let next $\Delta(\rho)$ be irreducible in F. Let

$$x_1 = X_1/f(\kappa), \quad y_1 = Y_1/f(\kappa^{-1}), \ldots, x_3 = X_3/f(\mu), \quad y_3 = Y_3/f(\mu^{-1}),$$

where $f(\rho)$ is a polynomial in ρ with coefficients in F. Then

$$x_1' = \kappa x_1, \quad y_1' = \kappa^{-1} y_1, \quad x_2' = \lambda x_2, \quad y_2' = \lambda^{-1} y_2, \quad x_3' = \mu x_3, \quad y_3' = \mu^{-1} y_3. \tag{91}$$

---

* To show that $R_{12} \neq 0$, we note that (74) gives, for $\rho = \kappa$,

$$g\gamma = (\kappa + \kappa^{-1} - \delta)[\kappa^2 + \kappa^{-2} - a(\kappa + \kappa^{-1}) - g\beta],$$

$$\therefore \; R_{12} = g^{-1}(\kappa - \kappa^{-1})(\delta - \kappa - \kappa^{-1})\{-3\kappa^2 - 3\kappa^{-2} + 2(a + \delta)(\kappa + \kappa^{-1}) + g\beta - a\delta - 4\}.$$

But $\Delta(\rho)$ has no quadratic or quartic factor in F.

Let $\phi(x) = R_{12}/f(x)f(x^{-1})$. Then, as in §20,

$$\sum_{i=1}^{3}\begin{vmatrix}\xi_i & \eta_i\\ \xi_i^* & \eta_i^*\end{vmatrix} = \frac{1}{\phi(x)}\begin{vmatrix}x_1 & y_1\\ x_1^* & y_1^*\end{vmatrix} + \frac{1}{\phi(\lambda)}\begin{vmatrix}x_2 & y_2\\ x_2^* & y_2^*\end{vmatrix} + \frac{1}{\phi(\mu)}\begin{vmatrix}x_3 & y_3\\ x_3^* & y_3^*\end{vmatrix}.$$

Let $x + x^{-1} = K$. Then $\phi(x) = (x - x^{-1})\psi(x)$, where

$$\psi(x) = g^{-1}\{g\gamma + (\delta - K)^2(2K - \alpha)\}/f(x)f(x^{-1}).$$

Further discussion is limited to the case when F is the $GF[p^n]$. Then $\Delta(\rho) = 0$ has the roots

$$x,\ x^{p^n},\ x^{p^{2n}},\ x^{p^{3n}},\ x^{p^{4n}},\ x^{p^{5n}} \qquad (x^{p^{6n}} = x).$$

Among these consequently occurs x^{-1}. Now $x^{-1} = x^{p^n}$ would give

$$x^{p^{2n}} = x^{-p^n} = x;$$

$x^{-1} = x^{p^{2n}}$ would give $x^{p^{4n}} = x$; $x^{-1} = x^{p^{4n}}$ would give $x = x^{p^{6n}} = x^{-p^{2n}}$, whence $x^{p^{4n}} = x$. Also $x^{-1} \neq x$, $x^{-1} \neq x^{p^{5n}}$. Hence $x^{-1} = x^{p^{3n}}$. Hence

$$f(x^{-1}) = [f(x)]^{p^{3n}}, \quad K = x + x^{-1} = \text{element of } GF[p^{3n}].$$

To make $\psi(x) = 1$, it suffices to take as $f(x)$ a root of

$$f^{p^{3n}+1} = \gamma + g^{-1}(\delta - K)^2(2K - \alpha),$$

all of whose roots f belong to the $GF[p^{6n}]$. Indeed, the second member is a mark $\neq 0$ of the $GF[p^{3n}]$, so that the power $p^{3n} - 1$ of it equals unity.

THEOREM.— *Within $SA(6, p^n)$, two transformations* (73) *having the same irreducible characteristic determinant are conjugate.*

27. Consider finally the type, obtained at the end of Case I, §21,

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0\\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0\\ 0 & \gamma g & \alpha & g & 1 & 0\\ 0 & 0 & -g^{-1} & 0 & 0 & 0\\ 0 & 0 & g^{-1}h & 0 & \epsilon & h\\ 0 & 0 & 0 & 0 & -h^{-1} & 0 \end{pmatrix}. \tag{92}$$

It has the characteristic determinant

$$\rho^6 - (\rho^5 + \rho)(\alpha + \delta + \epsilon) + (\rho^4 + \rho^2)(3 + \alpha\delta + \alpha\epsilon + \delta\epsilon - \gamma g - g^{-1}h) \\ - \rho^3(2\alpha + 2\delta + 2\epsilon + \alpha\delta\epsilon - g\gamma\epsilon - \delta g^{-1}h) + 1. \tag{93}$$

28. We first determine in which cases (92) is conjugate within G with a transformation (55). The conditions that (92) shall replace X and Y, defined by (33), by respectively τY and $-\tau^{-1}X+\sigma Y$, are

$$\tau A=-b\gamma^{-1},\quad \tau B=a\gamma+b\delta+c\gamma g,\quad \tau C=b+c\alpha-g^{-1}d+g^{-1}he,$$
$$\tau D=cg,\qquad \tau E=c+e\varepsilon-h^{-1}j,\qquad \tau J=eh;$$
$$\sigma A-\tau^{-1}a=-B\gamma^{-1},\quad \sigma B-\tau^{-1}b=A\gamma+B\delta+C\gamma g,\quad \tau D-\tau^{-1}d=Cg,$$
$$\tau C-\tau^{-1}c=B+C\alpha-g^{-1}D+g^{-1}hE,$$
$$\sigma E-\tau^{-1}e=C+E\varepsilon-h^{-1}J,\quad \sigma J-\tau^{-1}j=hE.$$

The first six conditions give

$$\left.\begin{aligned} &b=-\gamma\tau A,\quad c=g^{-1}\tau D,\quad e=h^{-1}\tau J,\quad j=-h\tau E+g^{-1}h\tau D+\varepsilon\tau J,\\ &a=\gamma^{-1}\tau B+\delta\tau A-\tau D,\quad d=-g\tau C-g\gamma\tau A+\alpha\tau D+\tau J.\end{aligned}\right\}\quad (94)$$

Substituting these values in the second set of six, we get

$$D=-A(\sigma-\delta),\quad \gamma gC=B(\sigma-\delta),\quad C=E(\sigma-\varepsilon),\quad hD=gJ(\sigma-\varepsilon),\quad (95)$$
$$hE=gC(\sigma-\alpha)-gB=B\{\gamma^{-1}(\sigma-\alpha)(\sigma-\delta)-g\},\quad (96)$$
$$J=D(\sigma-\alpha)+g\gamma A=A\{g\gamma-(\sigma-\alpha)(\sigma-\delta)\}.\quad (97)$$

Upon setting $\sigma=\rho+\rho^{-1}$, $(93)=0$ becomes

$$W(\sigma)\equiv g\gamma(\sigma-\varepsilon)+g^{-1}h(\sigma-\delta)-(\sigma-\alpha)(\sigma-\delta)(\sigma-\varepsilon)=0,\quad (98)$$

which is therefore the condition that $\rho^2-\sigma\rho+1$ shall be a factor of $\Delta(\rho)$. We assume that this necessary condition for the conjugacy is satisfied.

In view of $(95)_1$ and (97), $(95)_4$ reduces to $AW=0$. Likewise $(95)_3$ reduces to $BW=0$. Hence we may drop $(95)_3$ and $(95)_4$.

In view of relations (94)-(98), Abelian condition (34) becomes*

$$\frac{1}{\tau}=V(A^2+\sigma\gamma^{-1}AB+\gamma^{-2}B^2),$$
$$V\equiv\gamma+h^{-1}g^2\gamma^2+g^{-1}(\sigma-\delta)^2-2h^{-1}g\gamma(\sigma-\alpha)(\sigma-\delta)+h^{-1}(\sigma-\alpha)^2(\sigma-\delta)^2.$$

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* The only reduction made was in the coefficient of $\gamma^{-1}AB$, viz.,

$$P=\gamma\delta+g^{-1}(2\sigma-\alpha)(\sigma-\delta)^2+h^{-1}\varepsilon g^2\gamma^2-2\varepsilon h^{-1}g\gamma(\sigma-\alpha)(\sigma-\delta)+h^{-1}(\sigma-\alpha)^2(\sigma-\delta)^2,$$
$$\therefore\ P-\sigma V=\gamma(\delta-\sigma)+g^{-1}(\sigma-\alpha)(\sigma-\delta)^2+h^{-1}g^2\gamma^2(\varepsilon-\sigma)-2h^{-1}g\gamma(\sigma-\alpha)(\sigma-\delta)(\varepsilon-\sigma)$$
$$+h^{-1}(\sigma-\alpha)^2(\sigma-\delta)^2(\varepsilon-\sigma).$$

Eliminating the last term by using (98) multiplied by $h^{-1}(\sigma-\alpha)(\sigma-\delta)$, we find that

$$P-\sigma V=-h^{-1}g\gamma W=0.$$

If $V \neq 0$, the condition merely determines τ, whatever values, not both zero, in F be assigned to A and B. Now $V=0$ if, and only if, (98) has a double root. Indeed, (98) may be written

$$\sigma - \varepsilon = g^{-1}h(\sigma - \delta)/\{(\sigma - \alpha)(\sigma - \delta) - g\gamma\},$$

since the denominator $\neq 0$. Differentiating* and clearing of fractions, we obtain $V=0$. If (98) has a double root, not a triple root, the remaining root lies in F and leads to the conjugacy.

THEOREM.—*Within G, transformation* (92) *is conjugate with a type* (55) *if, and only if, its characteristic determinant* (93) *has in F an irreducible factor $\rho^2 - \sigma\rho + 1$ which is not a triple factor.*

29. Consider next (92) when (93) is either irreducible in F or is the product of two irreducible cubic factors (§25). Proceeding as in §26, we find that the only function, aside from a constant factor, which (92) multiplies by a constant κ is

$$X_1 = \xi_1 - \gamma\kappa\eta_1 + g^{-1}D\xi_2 + \kappa^{-1}D\eta_2 + \{\kappa\gamma g + (\kappa + \kappa^{-1} - \alpha)D\}(h^{-1}\xi_3 + \kappa^{-1}\eta_3),$$

where $D \equiv -\kappa^2 + \kappa\delta - 1$, and κ is a root of (93). Let $D' = -\lambda^2 + \lambda\delta - 1$. Then

$$\begin{aligned} R_{13} = &\begin{vmatrix} 1 & -\gamma\kappa \\ 1 & -\gamma\lambda \end{vmatrix} + \begin{vmatrix} g^{-1}D & \kappa^{-1}D \\ g^{-1}D' & g^{-1}D' \end{vmatrix} + \begin{vmatrix} h^{-1}[\kappa\gamma g + (\kappa + \kappa^{-1} - a)D] & \kappa^{-1}[\kappa\gamma g - (\kappa + \kappa^{-1} - a)D] \\ h^{-1}[\lambda\gamma g + (\lambda + \lambda^{-1} - a)D'] & \lambda^{-1}[\lambda\gamma g + (\lambda + \lambda^{-1} - a)D'] \end{vmatrix} \\ = &(\kappa - \lambda)\{\gamma + g^{-1}(\delta - k)(\delta - l) \\ &\qquad + h^{-1}[\gamma g - (\alpha - k)(\delta - k)][\gamma g - (\alpha - l)(\delta - l)]\}, \end{aligned}$$

where $k = \kappa + \kappa^{-1}$, $l = \lambda + \lambda^{-1}$, $m = \mu + \mu^{-1}$ are the roots of (98), viz.,

$$\begin{aligned} -W(\sigma) \equiv \sigma^3 - \sigma^2(\alpha + \delta + \varepsilon) + \sigma(\alpha\delta + \alpha\varepsilon + \delta\varepsilon - g\gamma - g^{-1}h) \\ - \alpha\delta\varepsilon + g\gamma\varepsilon + g^{-1}h\delta = 0. \qquad (98') \end{aligned}$$

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* That this argument is valid (here a great simplification) follows since, for $d \neq 0$, the system $\phi(\sigma) = 0$, $\phi'(\sigma) = 0$ is equivalent to

$$\phi/d = 0, \quad (d\phi' - \phi d')/d^2 = 0.$$

Hence $R_{13} = (x - \lambda) R$, where R is symmetrical in k and l, and hence expressible as a function of m:

$$R = \gamma - \frac{g^{-1} W(\delta)}{\delta - m} + h^{-1} g^2 \gamma^2 + h^{-1} \frac{W(\alpha)\, W(\delta)}{(\alpha - m)(\delta - m)}$$
$$- h^{-1} \gamma g \left[2\alpha\delta - (\alpha + \delta)(\alpha + \delta + \varepsilon - m) \right.$$
$$\left. + (\alpha + \delta + \varepsilon)^2 - 2(\alpha\delta + \alpha\varepsilon + \delta\varepsilon - g\gamma - g^{-1}h) - m^2 \right].$$

Now $$W(\delta) = g\gamma(\delta - \varepsilon), \quad W(\alpha) = g\gamma(\alpha - \varepsilon) + g^{-1}h(\alpha - \delta).$$

Then $h\gamma^{-1} g^{-1} (\alpha - m)(\delta - m) R$ gives a polynomial $m^4 - \ldots$. But m is a root of the cubic $(98')$. Multiplying it by m and subtracting from the preceding quartic, we obtain $(\alpha + \delta - \varepsilon)\, W(m)$. Hence $R_{13} = 0$. As in §26, equations (88) hold, while

$$R_{12} = (x - x^{-1}) \{ \gamma + g^{-1}(\delta - k)^2 + h^{-1} [\gamma g - (\alpha - k)(\delta - k)]^2 \},$$

where $k = x + x^{-1}$. Proceeding as in §27, we have the

THEOREM.—*Within the group $SA(6, F)$, two transformations (92) are conjugate if they have as common characteristic determinant the product of two irreducible cubic factors. Within $SA(6, p^n)$, two transformations (92) are conjugate if they have the same irreducible characteristic determinant.*

30. It remains to consider those transformations of the types (73), (75) and (92) whose characteristic determinants are the cube of an irreducible factor $\rho^2 - \sigma\rho + 1$ in F.

For (75) the conditions are $\delta = \alpha$, $g = -1$ (§23) and then $\Delta(\rho) = (\rho^2 - \alpha\rho + 1)^3$. Denote this (75) by S, and let $x^2 - \alpha x + 1 = 0$. The only functions which S multiplies by x are multiples of $X \equiv x\xi_2 - \xi_3$. If a function Y, independent of X, is replaced by $xY + tX$, then Y is a multiple of

$$\xi_1 - \gamma x \eta_1 + (t - xe)\,\xi_2 + e\xi_3, \quad t(\alpha - 2x) \equiv \gamma x.$$

Here $\alpha - 2x = x^{-1} - x \neq 0$. We take $e = 0$ and set $X_1 = Y$, $X_2 = lX$, where $l = tx^{-1}$. Then S replaces X_1 by $xX_1 + xX_2$. We seek four more functions such that the six shall be conjugate in pairs (the two of a pair being derived

from one another by replacing x by x^{-1}) and such that, when the six are properly ordered, the general Abelian conditions

$$R_{12} = R_{34} = R_{56} \neq 0\,, \quad R_{ij} = 0 \text{ [unless } (i, j) = (1,\ 2),\ (3,\ 4),\ (5,\ 6)]$$

shall hold. One finds the following solution:

$$X_1 = \xi_1 - \gamma x \eta_1 + l x \xi_2\,, \quad X_2 = l\,(x\xi_2 - \xi_3),$$
$$X_3 = -\gamma^{-1} a l \xi_1 + 2l\eta_1 + (a - 2x^{-1})\,\eta_2 + (xa - 2)\,\eta_3\,,$$
$$Y_1 = \xi_1 - \gamma x^{-1}\eta_1 + \bar{l} x^{-1}\xi_2\,, \quad Y_3 = \bar{l}\,(x^{-1}\xi_2 - \xi_3),$$
$$Y_2 = -\gamma^{-1} a \bar{l} \xi_1 + 2\bar{l}\eta_1 + (a - 2x)\,\eta_2 + (x^{-1} a - 2)\,\eta_3\,,$$

where, as above, $l = \gamma \div (a - 2x)$. Here X_1 and Y_1 are conjugate, X_2 and Y_3, X_3 and Y_2. For the transformation from ξ_i, η_i to X_i, Y_i we have every $R_{ij} = 0$ except

$$R_{12} = R_{34} = R_{56} = \gamma\,(x - x^{-1}).$$

The transformation S becomes, in the new variables,

$$\left.\begin{array}{lll} X_1' = xX_1 \quad + xX_2\ , & X_2' = xX_2\ , & X_3' = xX_3 \quad - xX_1 \ + \bar{\sigma} X_2 + \bar{\tau} Y_3, \\ Y_1' = x^{-1} Y_1 + x^{-1} Y_3, & Y_3' = x^{-1} Y_3, & Y_2' = x^{-1} Y_2 - x^{-1} Y_1 + \sigma Y_3 + \tau X_2, \end{array}\right\} \tag{100'}$$

where

$$\sigma = \frac{xa - 1 - a^2}{2x - a} + \gamma^{-1}\beta x^{-1}(a - 2x^{-1}), \quad \tau = \frac{xa - 1 - a^2}{a - 2x^{-1}} - \gamma^{-1}\beta x^{-1}(a - 2x),$$

so that $x\sigma + x^{-1}\bar{\sigma} = -1$. Transforming by

$$Y_2' = Y_2 + \lambda X_2, \quad X_3' = X_3 + \bar{\lambda} Y_3, \quad \tau + \lambda\,(x - x^{-1}) = 0,$$

we obtain a like transformation $S_{\sigma,\,x}$, with $\tau = \bar{\tau} = 0$. Let

$$x_i = X_i / f(x)\,, \quad y_i = Y_i / f(x^{-1}), \qquad (i = 1,\ 2,\ 3),$$

where $f(x) = a + bx$ is not identically zero. Then $S_{\sigma,\,x}$ retains its form

$$\left.\begin{array}{lll} x_1' = xx_1 \quad + xx_2\ , & x_2' = xx_2\ , & x_3' = xx_3 \quad - xx_1 \ + \bar{\sigma} x_2, \\ y_1' = x^{-1} y_1 + x^{-1} y_3, & y_3' = x^{-1} y_3, & y_2' = x^{-1} y_2 - x^{-1} y_1 + \sigma y_3. \end{array}\right\} \tag{100}$$

For the transformation T from ξ_i, η_i to x_i, y_i, we have

$$R_{12} = R_{34} = R_{56} = \frac{\gamma\,(x - x^{-1})}{f(x)\,f(x')}\,, \quad \text{remaining } R_{ij} = 0\,.$$

The condition $f(\varkappa)f(\varkappa^{-1}) \equiv a^2 + ab\alpha + b^2 = \gamma$ requires the solution of a quadratic in the field F. We here limit F to the $GF[p^n]$. Then $\varkappa^{-1} = \varkappa^{p^n}$, and the condition becomes $f^{p^n+1} = \gamma$, all of whose $p^n + 1$ roots f lie in the $GF[p^{2n}]$. Hence (§3, Corollary) for the resulting transformation T,

$$\sum_{i=1}^{3} \begin{vmatrix} \xi_i & \eta_i \\ \xi_i^* & \eta_i^* \end{vmatrix} = \frac{1}{\varkappa - \varkappa^{-1}} \sum_{i=1}^{3} \begin{vmatrix} x_i & y_i \\ x_i^* & y_i^* \end{vmatrix}, \tag{101}$$

independent of γ. Hence T belongs to $GA(6, p^{2n})$. One consequence is that $S_{\sigma,\kappa}$ belongs to $SA(6, p^{2n})$. Since $\varkappa\sigma + \varkappa^{-1}\bar{\sigma} = -1$, this may be verified directly.

We proceed to show that $S_{\Sigma,\kappa}$, where $\varkappa\Sigma + \varkappa^{-1}\bar{\Sigma} = -1$, is conjugate with $S_{\sigma,\kappa}$ within $SA(6, p^{2n})$. Let S be the general transformation (13) of §4. If $S_{\sigma,\kappa}S = SS_{\Sigma,\kappa}$, then

$$S = \begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 & 0 & 0 \\ 0 & \delta_{11} & 0 & 0 & 0 & \delta_{13} \\ 0 & 0 & \alpha_{11} & 0 & 0 & 0 \\ 0 & \delta_{21} & 0 & \delta_{11} & 0 & \delta_{23} \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \alpha_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{11} \end{pmatrix}, \tag{102}$$

where

$$\delta_{11}(\Sigma - \sigma) = \varkappa^{-1}(\delta_{13} + \delta_{21}), \quad \alpha_{11}(\bar{\Sigma} - \bar{\sigma}) = \varkappa(\alpha_{31} + \alpha_{12}). \tag{103}$$

The conjugacy of the variables requires that

$$\delta_{11} = \bar{\alpha}_{11}, \quad \delta_{13} = \bar{\alpha}_{12}, \quad \alpha_{31} = \bar{\delta}_{21}, \quad \alpha_{32} = \bar{\delta}_{23}. \tag{104}$$

The conditions that S shall be special Abelian are then

$$\alpha_{11}\bar{\alpha}_{11} = 1, \quad \delta_{21} = -\alpha_{12}/\alpha_{11}^2, \quad \delta_{21}\bar{\delta}_{21} + \bar{\alpha}_{11}\bar{\delta}_{23} + \alpha_{11}\delta_{23}. \tag{105}$$

The second condition (103) follows from the first, which becomes

$$\varkappa\Sigma - \varkappa\sigma = \alpha_{11}\bar{\alpha}_{12} - \bar{\alpha}_{11}\alpha_{12}. \tag{103$_1$}$$

For the $GF[p^n]$, this becomes, in view of $(105)_1$,

$$\left(\frac{\alpha_{12}}{\alpha_{11}}\right)^{p^n} - \frac{\alpha_{12}}{\alpha_{11}} = \varkappa\Sigma - \varkappa\sigma.$$

Raising both members to the power p^n, and applying

$$\varkappa\Sigma + \varkappa^{-1}\bar{\Sigma} = \varkappa\sigma + \varkappa^{-1}\bar{\sigma} = -1,$$

we get
$$\left(\frac{\alpha_{12}}{\alpha_{11}}\right)^{p^{2n}} - \left(\frac{\alpha_{12}}{\alpha_{11}}\right)^{p^n} = \kappa\sigma - \kappa\Sigma\,.$$

It follows by addition that α_{12}/α_{11} belongs to the $GF[p^{2n}]$, so that in this field $(103)_1$ can be solved for α_{12}/α_{11}. Likewise, $(105)_3$ can be solved for $\alpha_{11}\delta_{23}$ in terms of δ_{21} in the $GF[p^{2n}]$.

THEOREM.—*Within the group* $SA(6, p^n)$, *two transformations* (75) *having as common characteristic determinant the cube of an irreducible factor* $\rho^2 - \alpha\rho + 1$ *are conjugate. They are conjugate within* $GA(6, p^{2n})$ *with the canonical form* (100) *with* $\sigma = 1/(\kappa^{-1} - \kappa)$.

31. Consider next transformation (73) when its characteristic determinant (74) is the cube of a factor $\rho^2 - \sigma\rho + 1$ irreducible in F. As in §24,

$$3\sigma = \alpha + \delta, \quad \gamma g + (\delta - \sigma)^2(2\sigma - \alpha) = 0,$$
$$3\sigma^2 - 2\sigma\alpha - 2\sigma\delta + \alpha\delta - 2 - \beta g = 0. \tag{106}$$

Let $\kappa^2 - \sigma\kappa + 1 = 0$. The only function which (73) multiplies by κ is, aside from a constant factor, given by (85). It now becomes

$$X_2 = \xi_1 - \gamma\kappa\eta_1 + g^{-1}\kappa^2(\delta - \sigma)\xi_2$$
$$+ (\delta - \sigma)\eta_2 + g^{-1}(\delta - \sigma)(\alpha - \kappa^{-1})\xi_3 + \kappa(\delta - \sigma)\eta_3.$$

Replacing κ by κ^{-1}, we obtain the conjugate function Y_3. In the present notation, relation (89) gives, in view of $(106)_2$,

$$R_{36} = g^{-1}(\kappa - \kappa^{-1})\{\gamma g + (\delta - \sigma)^2(2\sigma - \alpha)\} = 0.$$

We seek a function X_1 such that, under transformation (73),

$$X_1' = \kappa X_1 + \kappa X_2.$$

It suffices* to take zero as the coefficient of η_1 in X_1. Then must

$$X_1 = -\xi_1 + g^{-1}(2\kappa - 2\kappa^2\sigma + \kappa^2\delta)\xi_2 + (2\kappa^{-1} - \delta)\eta_2$$
$$+ g^{-1}\kappa^{-1}(2\alpha + 2\delta - \sigma - \alpha\delta\kappa - 2\kappa^{-1})\xi_3 + (2 - \sigma\kappa)\eta_3.$$

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* Before attempting the complicated analysis of this section, I first carried out the indicated reduction for the field of order $p^n = 2$. Then must $g = \gamma = \alpha = \beta \equiv 1$, $\delta = 0$. Then (73) is reduced to the canonical form (100), with $\sigma = 1$, by the *special Abelian* transformation

$$X_1' = \xi_1 + \kappa^2\xi_3 + \kappa\eta_3, \qquad X_2' = \xi_1 + \kappa\eta_1 + \kappa^2\xi_2 + \eta_2 + \kappa\xi_3 + \kappa\eta_3,$$
$$Y_1' = \xi_1 + \kappa\xi_3 + \kappa^2\eta_3, \qquad Y_3' = \xi_1 + \kappa^2\eta_1 + \kappa\xi_2 + \eta_2 + \kappa^2\xi_3 + \kappa^2\eta_3,$$
$$Y_2' = \xi_1 + \eta_1 + \kappa\xi_2 + \xi_3 + \eta^3, \quad X_3' = \xi_1 + \eta_1 + \kappa^2\xi_2 + \xi_3 + \eta_3.$$

Denote its conjugate by Y_1. Then $R_{13}=0$. Let

$$X_3 = q\xi_1 + r\eta_1 + s\xi_2 + v\xi_3 + w\eta_3, \quad Y_2 = \overline{X}_3,$$

it sufficing to take zero as coefficients of η_2. We require that R_{35}, R_{15} and R_{25} shall vanish (i. e., the Abelian conditions connecting X_3 with X_2, X_1, Y_1 respectively):

$$\begin{aligned}
&r + \gamma\varkappa q - s(\delta-\sigma) + g^{-1}(\delta-\sigma)(\alpha-\varkappa^{-1})w - \varkappa(\delta-\sigma)v &&= 0,\\
&-r - s(2\varkappa^{-1}-\delta) + g^{-1}\varkappa^{-1}(2\alpha+2\delta-\sigma-\alpha\delta\varkappa-2\varkappa^{-1})w - (2-\sigma\varkappa)v &&= 0,\\
&-r - s(2\varkappa-\delta) + g^{-1}\varkappa(2\alpha+2\delta-\sigma-\alpha\delta\varkappa^{-1}-2\varkappa)w - (2-\sigma\varkappa^{-1})v &&= 0,
\end{aligned}$$

Subtracting the third from the second and dividing by $\varkappa - \varkappa^{-1}$, we get

$$2s + \sigma v - 3g^{-1}\sigma w = 0. \tag{107}$$

The second and first then give r and q in terms of s, w, v. Forming the function X_3' by which (73) replaces X_3, we obtain for $X_3' - \varkappa X_3 - \varkappa X_1$ [see desired canonical form (100)] the expression

$$\begin{aligned}
&(-\varkappa - \varkappa q - r\gamma^{-1})\xi_1 + (q\gamma + r\delta + w\gamma - \varkappa r)\eta_1 + \eta_2(w - \varkappa\delta + 2)\\
&\quad + [r + s\alpha + v + w\beta - \varkappa s + \varkappa g^{-1}(2\varkappa - 2\varkappa^2\sigma + \varkappa^2\delta)]\xi_2\\
&\quad + g^{-1}(w\alpha - g\varkappa v + 5\sigma - \alpha\delta\varkappa - 2\varkappa^{-1})\xi_3 + (sg - \varkappa w + 2\varkappa - \varkappa^2\sigma)\eta_3.
\end{aligned}$$

We seek the conditions under which this shall equal

$$\begin{aligned}
mX_2 + nY_3 \equiv\ &(m+n)\xi_1 - \gamma(m\varkappa + n\varkappa^{-1})\eta_1 + g^{-1}(\delta-\sigma)(m\varkappa^2 + n\varkappa^{-2})\xi_2\\
&+ (\delta-\sigma)(m+n)\eta_2 + g^{-1}(\delta-\sigma)[m(\alpha-\varkappa^{-1}) + n(\alpha-\varkappa)]\xi_3\\
&+ (\delta-\sigma)(\varkappa m + \varkappa^{-1}n)\eta_3.
\end{aligned}$$

Comparing the coefficients of ξ_1 and η_2, we get

$$(\delta-\sigma)(-\varkappa - \varkappa q - r\gamma^{-1}) = w - \varkappa\delta + 2,$$

Eliminating r and q by the above relations and, by $(106)_2$, also γ, we obtain, after dividing out $(\delta-\sigma)^2$, the condition

$$\phi \equiv -s - v\varkappa + g^{-1}(2\sigma - \varkappa^{-1})w - g^{-1}(\alpha - 2\sigma)(2 - \varkappa\sigma). \tag{108}$$

Comparing, similarly, η_1 and η_3, we get $(\sigma-\delta)(1-\varkappa\sigma)\phi = 0$. Solving the conditions from η_2 and η_3, we get

$$\left.\begin{aligned}
(\delta-\sigma)(\varkappa-\varkappa^{-1})n &= 2\varkappa w - sg - \varkappa^2\delta + \varkappa^2\sigma,\\
(\delta-\sigma)(\varkappa-\varkappa^{-1})m &= -\sigma w + sg - 2\varkappa^{-1} + 2\varkappa - \varkappa^2\sigma + \delta.
\end{aligned}\right\} \tag{109}$$

Equating the coefficients of ξ_3, we get $\phi = 0$. Equating those of ξ_2, we get $\kappa\phi = 0$. Hence the six conditions reduce to (108) and (109). Adding $2\phi = 0$ to (107) and dividing by $\kappa^{-1} - \kappa$, we get $(110)_1$, by means of which (108) reduces to $(110)_2$:

$$w = gv - 2\kappa(\alpha - 2\sigma), \quad s = \sigma v - 3g^{-1}(\alpha - 2\sigma)\sigma\kappa. \tag{110}$$

In view of these values, we get

$$\left.\begin{aligned} r &= v(\sigma^2 + \sigma\delta - \alpha\delta) - g^{-1}(\alpha - 2\sigma)(4\kappa + 3\sigma\kappa\delta - 2\alpha\delta\kappa), \\ \gamma\kappa q &= g^{-1}(\alpha - 2\sigma)(2\alpha - 4\kappa^{-1} - \sigma\kappa\alpha + \sigma\kappa\delta). \end{aligned}\right\} \tag{111}$$

Now the Abelian function for X_3, Y_3 is

$$\begin{aligned} R_{56} \equiv -\gamma\kappa^{-1}q - r + s(\delta - \sigma) + v\kappa^{-1}(\delta - \sigma) - wg^{-1}(\delta - \sigma)(\alpha - \kappa) \\ = g^{-1}(\alpha - 2\sigma)(\sigma^2 - 4)(\kappa^{-1} - \kappa). \end{aligned}$$

Now R_{34} (on X_2 and Y_2, conjugate to Y_3 and X_3 respectively) equals $-\overline{R}_{56}$, whence $R_{34} = R_{56}$. After some computation, we find that $R_{12} = R_{56}$ and that for the function for Y_2, X_3,

$$R_{45} \equiv \bar{q}r - \bar{r}q + \bar{v}w - \bar{w}v = (\sigma^2 - 4)(v - \bar{v}) - \tau(\kappa^{-1} - \kappa),$$

where

$$\tau \equiv \gamma^{-1}g^{-2}(\alpha - 2\sigma)^2(4 + 3\sigma\delta - 2\alpha\delta)(2\alpha\sigma - 4\sigma^2 + 4 - \alpha\sigma^2 + \delta\sigma^2).$$

Since $\sigma^2 - 4 = (\kappa - \kappa^{-1})^2 \neq 0$, and since $\overline{R}_{45} = -R_{45}$, we may choose v to make $R_{45} = 0$. We may, for example, take $(\sigma^2 - 4)v = \tau\kappa^{-1}$.

Making the transformation of variables (99), where now

$$f(\kappa)\,f(\kappa^{-1}) = -g^{-1}(\alpha - 2\sigma)(\sigma^2 - 4),$$

we have for the transformation T of variables from ξ_i, η_i to x_i, y_i,

$$R'_{12} = R'_{34} = R'_{56} = \kappa - \kappa^{-1}, \text{ the remaining } R'_{ij} = 0.$$

Hence (101) holds, so that T belongs to the $GA(6, p^{2n})$. The theorem of §30 therefore holds also for transformations (73).

32. It remains to consider type (92) when its characteristic determinant (93) is the cube of an irreducible factor $\rho^2 - \sigma\rho + 1$.

For the case when F is the $GF[2]$, $\sigma \equiv 1$, $\gamma = g = h = 1$, and

$$\alpha + \delta + \epsilon \equiv 1, \quad \alpha\delta + \alpha\epsilon + \delta\epsilon + 3 \equiv 0, \quad \alpha\delta\epsilon + \delta + \epsilon \equiv 1 \pmod 2.$$

By the first, $\alpha(\delta+\epsilon)\equiv\alpha^2+\alpha\equiv 0$, so that the second gives $\alpha\delta\equiv 1$. Hence $\alpha\equiv\delta\equiv\epsilon\equiv 1$. Let κ be a root of $\rho^2-\rho+1=0$. A function which (92) multiplies by κ is given by X_1 of §29; it is now written X_2:

$$X_2=\xi_1+\kappa\eta_1+\kappa\xi_3+\eta_3, \quad Y_3=\overline{X_2}=\xi_1+\kappa^2\eta_1+\kappa^2\xi_3+\eta_3.$$

Now $X_1'=\kappa X_1+\kappa X_2$, for $X_1=\xi_1+\ldots$ if, and only if,

$$X_1=\xi_1+\kappa\xi_2+\eta_2+\eta_3, \quad Y_1=\overline{X_1}=\xi_1+\kappa^2\xi_2+\eta_2+\eta_3.$$

Then $R_{12}=1$, $R_{13}=R_{16}=0$. Hence $R_{23}=R_{26}=0$. Let

$$X_3=q\xi_1+r\eta_1+s\xi_2+t\eta_2+v\xi_3+w\eta_3, \quad Y_2=\overline{X_3}.$$

The Abelian conditions connecting X_3 with X_1, Y_1, X_2, X_3 give

$$r+\kappa t+s+v=0, \qquad r+\kappa^2 t+s+v=0,$$
$$r+\kappa q+\kappa w+v=0, \quad r+\kappa^2 q+\kappa^2 w+v=1.$$

Hence $t=0$, $r+s+v=0$, $q+w=1$, $r+v=\kappa$, so that

$$X_3=(w+1)\xi_1+r\eta_1+\kappa\xi_2+(r+\kappa)\xi_3+w\eta_3.$$

Let our special (92) replace X_3 by X_3'. We find that

$$X_3'-\kappa X_3+\kappa X_1\equiv\kappa^2 X_2+(r+\kappa^2+\kappa w)\,Y_3.$$

Comparing with (100′), we have $\bar{\sigma}=\kappa^2$, whence $\kappa\sigma+\kappa^{-1}\bar{\sigma}=\kappa^2+\kappa=-1$. Now $R_{45}=r+\bar{r}+\kappa\bar{w}+\kappa^{-1}w$. To make $R_{45}=0$, we dispose of the undetermined r and w by setting $r=w=0$. The resulting transformation from ξ_i, η_i to X_i, Y_i is therefore a *special Abelian* transformation in the $GF[2^2]$, which transforms (92) into (100′) for $\sigma=\tau=\kappa$.

I refrain from the verification that, for the general $GF[p^n]$, transformation (92) with $\Delta(\rho)=(\rho^2-\sigma\rho+1)^3$ can be transformed into (100) by an Abelian transformation in the $GF[p^{2n}]$ such that relation (101) holds. Indeed, (92) and (100), with $\kappa^2-\sigma\kappa+1=0$, are conjugate under *linear* transformation in the $GF[p^{2n}]$, since the characteristic determinant* of each has the single invariant

* Give it the notation (c_{ij}) and denote the minor of c_{ij} by c_{ij}'. Then, for (100),

$$c_{35}'=(\kappa^{-1}-\rho)^3(\kappa^2+\bar{\sigma}\kappa-\bar{\sigma}\rho), \quad c_{64}'=(\kappa-\rho)^3(\kappa^{-2}+\sigma\kappa^{-1}-\sigma\rho),$$

which have no common factor involving ρ. Likewise, for (92),

$$c_{61}'=\gamma h\rho, \quad c_{41}'=\gamma g(\rho^2-\epsilon\rho+1).$$

The theorem holds true for (73), since $c_{41}'=\gamma g\rho$, $c_{21}'=\gamma(\rho^4-\rho^3 a-\rho^2 g\beta-\rho a+1)$.

factor $\Delta(\rho)$. Moreover, the conjecture gains weight if we note that types (73), (75), (77) and (92) have led to uniform results thus far, the separation of cases appearing to be not vital but only to avoid the impracticability of treating the parent type (72) subject to numerous lengthy Abelian conditions.

33. The classification in §21 led to four types, (73), (75), (77) and (92). Of these, (77) was reduced to (55) or (75) in §22. In §§23–32, the types (73), (75), (92) were treated separately, the conjugacies which occur in a given type being determined. It remains to determine the conjugacies occurring between the various transformations of the aggregate of the three types. This, however, requires no further computation, at least when we take F to be the $GF[p^n]$. Either a transformation of the aggregate is conjugate within $SA(6, p^n)$ with a type (55) or else it can be transformed into one of the canonical forms (91) and (100) by a general Abelian transformation T in the $GF[p^{nt}]$, where t is the common degree of the irreducible factors of $\Delta(\rho)$, the multiplier (μ of §2) of T being unity or $x - x^{-1}$, where $\Delta(x) = 0$.

THEOREM.— *Within* $SA(6, p^n)$, *two transformations, selected from the aggregate of the types* (73), (75), (77), (92), *but such that neither is conjugate with* (55), *are conjugate with each other, if, and only if, they have the same characteristic determinant.*

Types left for treatment by induction, §§34–40.

34. It remains to consider (16), (20), (23) and (55). Of these, (20) is the product of a special Abelian transformation on $\xi_1, \eta_1, \xi_2, \eta_2$ by one on ξ_i, η_i $(i>2)$, occurring only when $m \gtreqless 3$. Each of the other three is a product of a special Abelian transformation on ξ_1, η_1 by one on ξ_i, η_i $(i > 1)$.

35. Let first $m = 2$. Then (16) is the product of $T_{1,\kappa} T$, where T is a binary transformation of determinant unity on ξ_2, η_2. We apply to T the methods of §1. Let F be the $GF[p^n]$. The resulting types are $T_{1,\kappa} T_{2,\lambda}$, $T_{1,\kappa} T_{2,\pm 1} L_{2,\mu}$, $\mu = 1$ or a particular not-square ν, and $T_{1,\kappa} S_1$. We may replace S_1 by the ultimate canonical form $T_{2,\lambda}$, $\lambda^{p^n+1} = 1$, λ in the $GF[p^{2n}]$.

For (23) and (55) we similarly reduce not only the partial transformation on ξ_2, η_2, but also that on ξ_1, η_1.

Among the types arising for an arbitrary field F is $L_{1,\mu} L_{2,\tau}$, μ and τ in F. We proceed to investigate their conjugacy within G. Now

$$L_{1,\mu} L_{2,\tau}, S = S L_{1,\rho} L_{2,\sigma},$$

where S is a general quaternary transformation, if and only if,

$$\rho\beta_{11}=\rho\beta_{12}=0,\quad \sigma\beta_{21}=\sigma\beta_{22}=0,\quad \rho\delta_{11}=\mu\alpha_{11},\quad \rho\delta_{12}=\tau\alpha_{12},$$
$$\mu\beta_{11}=\mu\beta_{21}=0,\quad \tau\beta_{12}=\tau\beta_{22}=0,\quad \sigma\delta_{22}=\tau\alpha_{22},\quad \sigma\delta_{21}=\mu\alpha_{21}.$$

If $\tau=0$, $\mu\neq 0$, $\sigma\neq 0$, then β_{11}, β_{21}, β_{22}, δ_{22} are zero and

$$\rho\beta_{12}=0,\quad \rho\delta_{12}=0,\quad \sigma\delta_{21}=\mu\alpha_{21}.$$

Then $\alpha_{21}\delta_{21}=1$, by an Abelian condition, whence $\sigma/\mu=\alpha_{21}^2$. Also $\rho=0$, the determinant of S not vanishing. Likewise if $\tau=\sigma=0$, $\mu\neq 0$, $\rho\neq 0$, then $\beta_{11}=\beta_{12}=\delta_{12}=0$, $\alpha_{11}\delta_{11}=1$, $\rho\delta_{11}=\mu\alpha_{11}$, so that $\rho/\mu=\alpha_{11}^2$. *Hence the only transformations* $L_{1,\rho}L_{2,\sigma}$ *conjugate with* $L_{1,\mu}$ *are* $L_{1,\mu\alpha^2}$ *and* $L_{2,\mu\alpha^2}$.

Lastly, if μ, τ, ρ, σ are all $\neq 0$,

$$S=\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \rho^{-1}\mu\alpha_{11} & 0 & \rho^{-1}\tau\alpha_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \sigma^{-1}\mu\alpha_{21} & 0 & \sigma^{-1}\tau\alpha_{22} \end{pmatrix}.$$

The Abelian conditions reduce to the following

$$\mu\alpha_{11}^2+\tau\alpha_{12}^2=\rho,\quad \mu\alpha_{21}^2+\tau\alpha_{22}^2=\sigma,\quad \mu\alpha_{11}\alpha_{21}+\tau\alpha_{12}\alpha_{22}=0, \tag{112}$$

$$\alpha_{11}\gamma_{21}-\alpha_{21}\gamma_{11}+\alpha_{12}\gamma_{22}-\alpha_{22}\gamma_{12}=0. \tag{113}$$

If the α_{ij} can be chosen in F to satisfy (112), then the γ_{ij} can be chosen to satisfy (113). It is possible to take $\alpha_{11}=0$ if and only if ρ/τ and σ/μ are squares in F. Let next $\alpha_{11}\neq 0$, so that $\alpha_{22}\neq 0$, and set $\alpha_{12}=\varkappa\alpha_{11}$, $\alpha_{21}=\lambda\alpha_{22}$. Conditions (112) then become

$$\alpha_{11}^2(\mu+\tau\varkappa^2)=\rho,\quad \alpha_{22}^2(\tau+\mu\lambda^2)=\sigma,\quad \mu\lambda+\tau\varkappa=0. \tag{112'}$$

Multiplying the first by τ and the second by μ, and applying the third, we see that $\tau\rho/\sigma\mu$ must be a square in F. Let this condition be satisfied, so that the second condition (112′) will determine α_{22} in F when the first determines α_{11} in F. Consider, therefore, the first and third conditions only. If μ/ρ is a square, we may take $\varkappa=\lambda=0$. Let next μ/ρ be a not-square in F. If $\varkappa$ can be determined in F such that

$$\mu/\rho+\varkappa^2\tau/\rho=\text{square}, \tag{114}$$

then α_{11} and λ can be determined in F to satisfy the first and third conditions (112′). If τ/ρ is a square, $\tau\rho$ is a square, so that $\sigma\mu$ is a square by hypothesis

and consequently also μ/σ; but in this case we found that solutions were given by taking $\alpha_{11}=0$.

If μ, τ, ρ, σ *are each not zero*, $L_{1,\mu}L_{2,\tau}$ *and* $L_{1,\rho}L_{2,\sigma}$ *are conjugate within* G *except in the cases* (i), $\tau\rho/\sigma\mu$ *is a not-square in* F; (ii), $\tau\rho/\sigma\mu$ *is a square, while* μ/ρ, τ/ρ *and* $\mu/\rho+x^2\tau/\rho$ *are not-squares for every* x *in* F.

For the $GF[p^n]$, $p>2$, $\mu/\rho+x^2\tau/\rho$ can be made a square (or a not-square) by choice of x, so that the single type of the form $L_{1,\nu_i}L_{2,\nu_j}$ is conjugate with $L_{1,1}L_{2,1}$. For the field of all real numbers, the not-squares are negative so that for case (ii) $\mu/\rho+x^2\tau/\rho$ is a not-square for every x; hence no type $L_{1,\nu_i}L_{2,\nu_j}$ is conjugate with $L_{1,1}L_{2,1}$ or with $L_{1,1}L_{2,\nu_k}$.

36. Summary of all canonical forms for $m=2$ and F the $GF[p^n]$. Let $\mu=1$ or a particular not-square ν if $p>2$, $\mu=1$ if $p=2$. The canonical forms with all elements in the $GF[p^n]$ are

$$T_{1,x}T_{2,\lambda};\quad T_{1,x}T_{2,\pm 1}L_{2,\mu};\quad L_{1,1}L_{2,\mu}T_{1,\pm 1}T_{2,\pm 1};\ (19)\text{ with } x\neq x^{-1};$$

$$\text{additional for } p>2:\ L_{1,1}T_{1,-1}L_{2,\mu};\ L_{1,\nu}T_{1,-1}L_{2,\mu};\ A_{\mu}T_{1,\pm 1}T_{2,\pm 1}=(25);$$

$$\text{additional for } p=2:\ R_{\beta}=(26);\ R_{1,2,1}=(19)\text{ for } x=1.$$

The canonical forms with not all elements in the $GF[p^n]$ are

(63) with $x^{p^n+1}=1$, $x^2\neq 1$; (68) with x, λ in the $GF[p^{2n}]$ or $GF[p^{4n}]$;

$T_{1,x}T_{2,\lambda}$ with $x^{p^n-1}=1$, $\lambda^{p^n+1}=1$, $\lambda^2\neq 1$; $T_{1,x}T_{2,\lambda}$ with $x^{p^n+1}=\lambda^{p^n+1}$, $x^2\neq 1$, $\lambda^2\neq 1$; $L_{1,\mu}T_{1,\pm 1}T_{2,\lambda}$ with $\lambda^{p^n+1}=1$, $\lambda^2\neq 1$.

From this list we obtain a list of canonical forms no two of which are conjugate by retaining but one of the set

$$T_{1,a^{\pm 1}}T_{2,b},\quad T_{1,a^{\pm 1}}T_{2,b^{-1}},\quad T_{1,b}T_{2,a^{\pm 1}},\quad T_{1,b^{-1}}T_{2,a^{\pm 1}}.$$

37. Consider next (16) for $m=3$ and F the $GF[p^n]$. Changing the notation, we write it $T_{3,\omega}S$, where ω is a mark $\neq 0$ of F such that $\omega\neq\omega^{-1}$, and S is a special Abelian substitution on ξ_1, η_1, ξ_2, η_2. We give to S in turn the canonical forms of §36, except (19), which is treated in §38. However each form is subdivided as far as convenient in determining all the transformations of G commutative with $T_{3,\omega}S$. We therefore follow the order (but with the present notation) of the expanded list in Transactions, vol. 2 (1901), p. 132.

Type $T_{1,x}T_{2,\lambda}T_{3,\omega}$, x, x^{-1}, λ, λ^{-1}, ω, ω^{-1} all different.

Transforming by a suitable product of the M_i and P_{ij}, we can bring any one of the 6 multipliers into the first place and hence its inverse into the second place, any one of the remaining 4 into the third place, etc. Hence there are

$$\tfrac{1}{48}(p^n-3)(p^n-5)(p^n-7) \text{ or } \tfrac{1}{48}(2^n-2)(2^n-4)(2^n-6)$$

non-conjugate forms of this type, according as $p>2$ or $p=2$. A form is commutative only with the $(p^n-1)^3$ transformations $T_{1,a}T_{2,b}T_{3,c}$.

Type $T_{1,\kappa}T_{2,\kappa^{-1}}T_{3,\omega}$, κ, κ^{-1}, ω, ω^{-1} all distinct. There are

$$\tfrac{1}{4}(p^n-3)(p^n-5) \text{ or } \tfrac{1}{4}(2^n-2)(2^n-4)$$

non-conjugate forms of this type. Each is commutative only with $AT_{3,\tau}$, where A is the second matrix on p. 108 of Transactions. The number of commutative transformations is

$$(p^{2n}-1)(p^{2n}-p^n)(p^n-1).$$

Type $T_{1,\omega}T_{2,\omega}T_{3,\omega}$, $\omega \neq \omega^{-1}$. The number of non-conjugates is

$$\tfrac{1}{2}(p^n-3) \text{ or } \tfrac{1}{2}(2^n-2).$$

By a general theorem (Linear Groups, pp. 229–233), which is used repeatedly here to avoid computation, the type is commutative only with

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 & \alpha_{13} & 0 \\ 0 & \delta_{11} & 0 & \delta_{12} & 0 & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{22} & 0 & \alpha_{23} & 0 \\ 0 & \delta_{21} & 0 & \delta_{22} & 0 & \delta_{23} \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \alpha_{33} & 0 \\ 0 & \delta_{31} & 0 & \delta_{32} & 0 & \delta_{33} \end{pmatrix}. \tag{115}$$

Its determinant equals $|\alpha_{ij}|.|\delta_{ij}|$. The Abelian conditions are

$$\alpha_{i1}\delta_{i1}+\alpha_{i2}\delta_{i2}+\alpha_{i3}\delta_{i3}=1, \quad \alpha_{i1}\delta_{j1}+\alpha_{i2}\delta_{j2}+\alpha_{i3}\delta_{j3}=0,$$
$$(i,j=1,\ 2,\ 3;\ i\neq j).$$

Then $|\alpha_{ij}|.|\delta_{ij}|=1$. Hence the δ_{ij} are uniquely determined in terms of the α_{ij}. The group of the commutative substitutions is therefore simply isomorphic with the ternary general linear homogeneous group (Linear Groups, p. 77) of order

$$(p^{3n}-1)(p^{3n}-p^n)(p^{3n}-p^{2n}).$$

Type $T_{1,\kappa}T_{2,\pm1}T_{3,\omega}$, κ, κ^{-1}, ω, ω^{-1} all distinct. There are

$$\tfrac{1}{4}(p^n-3)(p^n-5) \text{ or } \tfrac{1}{8}(2^n-2)(2^n-4)$$

non-conjugate forms. Each is commutative only with

$$\xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1, \quad \xi_2' = e\xi_2 + f\eta_2, \quad \eta_2' = g\xi_2 + h\eta_2,$$
$$\xi_3' = b\xi_3, \quad \eta_3' = b^{-1}\eta_3, \qquad (116)$$

where $eh - fg = 1$. There are $(p^n - 1)^2(p^{2n} - 1)p^n$ of these.

Type $T_{1,\omega^{-1}}T_{2,\pm1}T_{3,\omega}$, $\omega \neq \omega^{-1}$. The number of non-conjugates is

$$p^n - 3 \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with AB, A as above and B any binary transformation of determinant 1 on ξ_2, η_2, giving

$$(p^{2n} - 1)(p^{2n} - p^n) \,.\, (p^{2n} - 1)p^n.$$

Type $T_{1,\kappa}T_{2,\pm1}L'_{2,\mu}T_{3,\omega}$, κ, κ^{-1}, ω, ω^{-1} all distinct, μ as in §36. The number of non-conjugate forms is

$$\tfrac{1}{2}(p^n - 3)(p^n - 5) \text{ or } \tfrac{1}{8}(2^n - 2)(2^n - 4).$$

Each is commutative only with (116) with $f = 0$, $e = h$, giving

$$2p^n(p^n - 1)^2 \text{ or } 2^n(2^n - 1)^2.$$

Type $T_{1,\omega^{-1}}T_{2,\pm1}L'_{2,\mu}T_{3,\omega}$. The number of non-conjugates is

$$2(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with $AT_{2,\pm1}L'_{2,g}$, A as above, giving

$$2p^n \,.\, (p^{2n} - 1)(p^{2n} - p^n) \text{ or } 2^n \,.\, (2^{2n} - 1)(2^{2n} - 2^n).$$

Type $T_{1,\pm1}T_{2,\pm1}T_{3,\omega}$, $\omega \neq \omega^{-1}$. The number of non-conjugates is

$$p^n - 3 \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative with $p^{4n}(p^{4n} - 1)(p^{2n} - 1)(p^n - 1)$ transformations.

Type $L_{1,\mu}T_{1,\pm1}T_{2,\pm1}T_{3,\omega}$. The number of non-conjugates is

$$2(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

By Trans., bottom of p. 114, each is commutative with only

$$2p^{4n}(p^{2n} - 1) \,.\, (p^n - 1) \text{ or } 2^{4n}(2^{2n} - 1) \,.\, (2^n - 1).$$

Type $L_{1,1}L_{2,1}T_{1,\pm1}T_{2,\pm1}T_{3,\omega}$. The number of non-conjugates is

$$p^n - 3 \text{ or } \tfrac{1}{2}(2^n - 2).$$

If $\varepsilon = \pm 1$ according as $p^n = 4l \pm 1$, there are

$$2p^{3n}(p^n - \varepsilon).(p^n - 1) \text{ or } 2^{4n}.(2^n - 1)$$

transformations commutative with each (Trans., p. 115).

Type $L_{1,1} L_{2,\nu} T_{1,\pm 1} T_{2,\pm 1} T_{3,\omega}$, for $p > 2$. There are $p^n - 3$ non-conjugate types, each commutative with only

$$2p^{3n}(p^n + \varepsilon).(p^n - 1).$$

Type $R_{1,2,1} T_{3,\omega}$, $p = 2$. Each of the $\frac{1}{2}(2^n - 2)$ non-conjugate forms is commutative with $2^{4n}(2^{2n} - 1).(2^n - 1)$ transformations (Tr., p. 111).

Type $R_\beta T_{3,\omega}$, $p = 2$. Each of the $2^n - 2$ non-conjugate forms is commutative with only $2.2^{2n}.(2^n - 1)$ transformations (Tr., p. 112).

Type $A_\mu T_{1,\pm 1} T_{2,\pm 1} T_{3,\omega}$, $p > 2$. Each of the $2(p^n - 3)$ non-conjugate forms is commutative with $2p^{2n}.(p^n - 1)$ transformations (Tr., p. 113).

Type $T_{1,a} T_{2,\lambda} T_{3,\omega}$, a, a^{-1}, ω, ω^{-1}, all distinct marks of the $GF[p^n]$, $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$. The number of non-conjugate forms is

$$\tfrac{1}{16}(p^n - 1)(p^n - 3)(p^n - 5) \text{ or } \tfrac{1}{16} 2^n (2^n - 2)(2^n - 4).$$

Each is commutative with the $(p^n - 1)^2(p^n + 1)$ $T_{1,c} T_{2,\lambda'} T_{3,d}$.

Type $T_{1,\omega} T_{2,\lambda} T_{3,\omega}$, ω and λ as before. There are

$$\tfrac{1}{4}(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms, each commutative with only

$$(p^{2n} - 1)(p^{2n} - p^n).(p^n + 1).$$

Type $T_{1,\pm 1} T_{2,\lambda} T_{3,\omega}$, ω and λ as before. There are

$$\tfrac{1}{2}(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms, each commutative with

$$p^n (p^{2n} - 1)(p^n + 1).(p^n - 1).$$

Type $T_{1,\pm 1} L_{1,\mu} T_{2,\lambda} T_{3,\omega}$, μ, λ, ω as before. There are

$$(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms, each commutative (Tr., p. 119) with

$$2p^n(p^n + 1).(p^n - 1) \text{ or } 2^n(2^n + 1).(2^n - 1).$$

Type $T_{1,\sigma} T_{2,\sigma^{p^n}} T_{3,\omega}$, $\sigma^{p^{2n}+1} = 1$, $\sigma^2 \neq 1$. There are

$$\tfrac{1}{8}(p^{2n}-1)(p^n-3) \text{ or } \tfrac{1}{8} 2^{2n}(2^n-2)$$

non-conjugate types (Tr., §18), each commutative with

$$(p^{2n}+1)(p^n-1).$$

Type $T_{1,\sigma} T_{2,\sigma^{p^n}} T_{3,\omega}$, $\sigma^{p^{2n}-1} = 1$, $\sigma^{p^n \pm 1} \neq 1$. There are

$$\tfrac{1}{8}(p^n-1)^2.(p^n-3) \text{ or } \tfrac{1}{8} 2^n(2^n-2).(2^n-2)$$

non-conjugate types (Tr., §19), each commutative with

$$(p^{2n}-1)(p^n-1).$$

Type $T_{1,\kappa} T_{2,\lambda} T_{3,\omega}$, $\kappa^{p^n+1} = \lambda^{p^n+1} = 1$, $\kappa^2 \neq 1$, $\lambda^2 \neq 1$, $\lambda \neq \kappa, \kappa^{-1}$. There are

$$\tfrac{1}{16}(p^n-1)(p^n-3).(p^n-3) \text{ or } \tfrac{1}{16} 2^n(2^n-2).(2^n-2)$$

non-conjugate forms (Tr., §20), each commutative with

$$(p^n+1)^2(p^n-1).$$

Type $T_{1,\kappa} T_{2,\kappa} T_{3,\omega}$, $\kappa^{p^n+1} = 1$, $\kappa^2 \neq 1$. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4} 2^n(2^n-2)$$

non-conjugate forms (Tr., §21), each commutative with

$$(p^n+1)p^n(p^{2n}-1).(p^n-1).$$

Type (63) $T_{3,\omega}$, $\kappa^{p^n+1} = 1$, $\kappa^2 \neq 1$. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4} 2^n(2^n-2)$$

non-conjugate forms (Tr., §22), each commutative with

$$p^n(p^n+1).(p^n-1).$$

Type $T_{1,-1} T_{3,\omega}$, $p > 2$. Each of the $\tfrac{1}{2}(p^n-3)$ non-conjugate forms is commutative (Tr., p. 109) with $[p^n(p^{2n}-1)]^2(p^n-1)$.

Types $L_{1,\mu} T_{1,-1} T_{3,\omega}$ and $L_{2,\mu} T_{1,-1} T_{3,\omega}$, $p > 2$. Each of the $2(p^n-3)$ non-conjugate forms is commutative with (Tr., §9)

$$2p^{2n}(p^{2n}-1).(p^n-1).$$

Types $L_{1,1} T_{1,-1} L_{2,\mu} T_{3,\omega}$ and $L_{1,\nu} T_{1,-1} L_{2,\mu} T_{3,\omega}$, $p > 2$. Each of the $2(p^n-3)$ non-conjugate forms is commutative with only $4p^{2n}(p^n-1)$ transformations (Tr., §10).

38. We pass to transformation (20) for $m = 3$ and F the $GF[p^n]$. We give the binary transformation of determinant unity on ξ_3, η_3 one of the canonical form of §1.

Type (19) $T_{3,\alpha}$, $\varkappa$, $\varkappa^{-1}$, α, α^{-1} all distinct marks of the $GF[p^n]$. Note that P_{12} transforms (19) into a similar transformation with $\varkappa^{-1}$ in place of $\varkappa$. Hence there are

$$\tfrac{1}{4}(p^n - 3)(p^n - 5) \text{ or } \tfrac{1}{4}(2^n - 2)(2^n - 4)$$

non-conjugate forms of our type. Each is commutative only with

$$\begin{array}{c|c} \Sigma & O \\ \hline O & \begin{matrix} e & o \\ o & e^{-1} \end{matrix} \end{array}, \quad \Sigma \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & ba^{-2} & 0 \\ 0 & 0 & a^{-1} & 0 \\ b & 0 & 0 & a \end{pmatrix}, \tag{117}$$

where O denotes a matrix all of whose elements are zero. The number of transformations is thus $p^n(p^n - 1)^2$.

Type (19) $T_{3,\kappa}$. The number of non-conjugates is

$$\tfrac{1}{2}(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with the linear transformations*

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & b_1 & 0 & 0 & c_1 \\ 0 & 0 & a_1 & 0 & 0 & 0 \\ b & 0 & 0 & a & c & 0 \\ d & 0 & 0 & 0 & e & 0 \\ 0 & 0 & d_1 & 0 & 0 & e_1 \end{pmatrix}. \tag{118}$$

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* This result may be verified directly or derived from the general theory, first rearranging the variables. Then the first transformation below is commutative only with the second :

$$\begin{array}{r|cccccc} & \xi_1 & \eta_2 & \xi_3 & \xi_2 & \eta_1 & \eta_3 \\ \hline \xi_1' = & \kappa & 0 & 0 & 0 & 0 & 0 \\ \eta_2' = & \kappa & \kappa & 0 & 0 & 0 & 0 \\ \xi_3' = & 0 & 0 & \kappa & 0 & 0 & 0 \\ \xi_2' = & 0 & 0 & 0 & \kappa^{-1} & 0 & 0 \\ \eta_1' = & 0 & 0 & 0 & \kappa^{-1} & \kappa^{-1} & 0 \\ \eta_3' = & 0 & 0 & 0 & 0 & 0 & \kappa^{-1} \end{array} \qquad \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ b & a & c & 0 & 0 & 0 \\ d & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & b_1 & a_1 & c_1 \\ 0 & 0 & 0 & d_1 & 0 & e_1 \end{pmatrix}.$$

The Abelian conditions are

$$\left.\begin{array}{ll} aa_1=1,\quad ee_1=1, & -a_1b+b_1a-c_1c=0, \\ -a_1d-ec_1=0, & -ad_1+ce_1=0, \end{array}\right\} \tag{118'}$$

whence

$$a_1=\frac{1}{a}, \quad e_1=\frac{1}{e}, \quad d_1=\frac{c}{ae}, \quad c_1=\frac{-d}{ae}, \quad b_1=\frac{b}{a^2}-\frac{cd}{a^2e}.$$

Hence c, b, d, a, e are arbitrary, with $a \neq 0$, $e \neq 0$, so that there are exactly $p^{3n}(p^n-1)^2$ commutative transformations.

Type (19) $T_{3,\pm 1}$. Each of the p^n-3 or $\frac{1}{2}(2^n-2)$ non-conjugate forms is commutative only with the transformations

$$\begin{array}{c|c} \Sigma & O \\ \hline O & \begin{matrix} rs \\ tu \end{matrix} \end{array}, \quad ru-ts=1, \tag{119}$$

Σ as in (117). There are $p^n(p^n-1)\cdot p^n(p^{2n}-1)$ of these.

Type (19) $T_{3,\pm 1}L'_{3,\mu}$. The number of non-conjugates is

$$2(p^n-3) \text{ or } \tfrac{1}{2}(2^n-2).$$

Each is commutative only with (119) for $s=0$, $r=u$, giving

$$2p^n.p^n(p^n-1) \text{ or } 2^n.2^n(2^n-1).$$

Type (19) $T_{3,\lambda}$, $\lambda^{p^n+1}=1$, $\lambda^2 \neq 1$. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4}2^n(2^n-2)$$

non-conjugate forms. Each is commutative only with (117) for $e^{p^n+1}=1$, giving $p^n(p^n-1)(p^n+1)$ transformations.

39. Consider next transformation (23) for $m=3$ and F the $GF[p^n]$. Changing the notation of the variables, we have $WT_{3,\pm 1}L'_{3,\tau}$ where W is a special Abelian transformation on $\xi_1, \eta_1, \xi_2, \eta_2$. Let the characteristic determinant of W be $D(\rho)$. The case when $D(\rho)$ has in F a root $x \neq x^{-1}$ was excluded by hypothesis (§8). We first suppose that every root of $D(\rho)$ is a root of $x^2=1$. Then W may be taken to be $R_{1,2,1}$, R_β, $A_\mu T_{1,\pm 1}T_{2,\pm 1}$ or a product of certain $L_{i,a}$,

$T_{i,-1}$, $i=1, 2$. In the last case, we replace $L'_{3,\tau}$ by its conjugate $L_{3,\tau}$ and have the following preliminary types:*

$$\begin{gathered} I;\ T_{3,-1};\ L_{1,\mu};\ L_{1,\mu}T_{1,-1};\ L_{2,\mu}T_{1,-1};\ L_{1,\mu}L_{2,1};\ L_{1,\mu}T_{1,-1}L_{2,\lambda}; \\ L_{1,\mu}T_{1,-1}L_{2,1}T_{2,-1};\ L_{1,\lambda}T_{1,-1}L_{2,1}L_{3,\mu};\ L_{1,1}L_{2,1}L_{3,\mu}; \\ (\lambda=\mu=1 \text{ if } p=2;\ \lambda,\ \mu=1,\ \nu \text{ if } p>2), \end{gathered} \tag{120}$$

together with their products by $T \equiv T_{1,-1}T_{2,-1}T_{3,-1}$.

For $p>2$, $L_{1,1}L_{2,1}L_{3,1}$ is not conjugate with† $L_{1,1}L_{2,1}L_{3,\nu}$ since

$$L_{1,1}L_{2,1}L_{3,1}S = SL_{1,\nu}L_{2,\nu}L_{3,\nu}$$

requires that, when S is given the form (13),

$$\beta_{ij}=0, \quad \alpha_{ij}=\nu\delta_{ij} \quad (i, j=1, 2, 3).$$

The Abelian conditions on S then reduce to

$$\delta_{1i}^2+\delta_{2i}^2+\delta_{3i}^2=1/\nu, \quad \delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}+\delta_{3i}\delta_{3j}=0 \quad (i, j=1, 2, 3;\ i \neq j),$$

together with linear homogeneous conditions on the γ_{ij}. The conditions on the δ_{ij} are precisely the conditions that the ternary transformation (δ_{ij}) shall replace $\xi_1^2+\xi_2^2+\xi_3^2$ by $\frac{1}{\nu}(\xi_1^2+\xi_2^2+\xi_3^2)$. But such a replacement is impossible‡ if ν is a not-square.

Next, $L_{1,1}$ and $L_{1,\nu}$ are not conjugate within G. For the conditions for $L_{1,1}(13)=(13)L_{1,\nu}$ are

$$\beta_{11}=\beta_{12}=\beta_{21}=\beta_{31}=\beta_{13}=\alpha_{21}=\alpha_{31}=\delta_{12}=\delta_{13}=0, \quad \alpha_{11}=\nu\delta_{11}$$

Then an Abelian condition gives $\alpha_{11}\delta_{11}=1$, whence $\nu=\alpha_{11}^2$.

That the two forms of the type $L_{1,\mu}L_{2,1}$ are not conjugate within G follows from the enumeration below of the commutative transformations.

That no two of the three types $L_{1,\mu}$, $L_{1,a}L_{2,b}$, $L_{1,c}L_{2,d}L_{3,e}$ are conjugate follows from the theory of canonical forms. Also, by the invariance of the roots of the characteristic equation, two conjugate forms (120) must contain the same number of factors $T_{i,-1}$, if $p>2$.

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* Noting that, by §35, $L_{i,1}L_{j,1}$ and $L_{i,\nu}L_{j,\nu}$ are conjugate. † Ib.

‡ Jordan, Traité, p. 171, for the case of modulus p and $m=3$. I have a direct proof of the following theorem for an arbitrary field F. No linear transformation with coefficients in F multiplies $\mu_1\xi_1^2+\mu_2\xi_2^2+\ldots+\mu_m\xi_m^2$ by a not-square in F, when m is odd.

If, for $p > 2$, S transforms $L_{1,\lambda}T_{1,-1}L_{2,1}L_{3,1}$ into $L_{1,\lambda}T_{1,-1}L_{2,1}L_{3,\nu}$, whose p^{th} powers equal $T_{1,-1}$, then S is the product of a special Abelian transformation on ξ_1, η_1 by one on $\xi_2, \eta_2, \xi_3, \eta_3$. The latter would transform $L_{2,1}L_{3,1}$ into $L_{2,1}L_{3,\nu}$, contrary to §33. Similarly, if S transforms $L_{1,\mu}T_{1,-1}L_{2,1}$ into $L_{1,\lambda}T_{1,-1}L_{2,\nu}$, it would transform $L_{2,1}$ into $L_{2,\nu}$.

Treating the remaining cases similarly, we find that no two of the transformations (120) are conjugate within G.

Type $L_{1,1}L_{2,1}L_{3,\mu}T$. It is commutative with (13) only if

$$\delta_{11}=\alpha_{11},\quad \delta_{12}=\alpha_{12},\quad \delta_{13}=\mu\alpha_{13},\quad \delta_{21}=\alpha_{21},\quad \beta_{ij}=0,\qquad (i,j=1,2,3),$$
$$\delta_{22}=\alpha_{22},\quad \delta_{33}=\alpha_{33},\quad \delta_{23}=\mu\alpha_{23},\quad \alpha_{31}=\mu\delta_{31},\quad \alpha_{32}=\mu\delta_{32}.$$

The Abelian conditions on (13) then reduce to

$$\left.\begin{array}{l}\alpha_{11}^2+\alpha_{12}^2+\mu\alpha_{13}^2=1,\quad \alpha_{21}^2+\alpha_{22}^2+\mu\alpha_{23}^2=1,\quad \alpha_{31}^2+\alpha_{32}^2+\mu\alpha_{33}^2=\mu;\\ \alpha_{i1}\alpha_{j1}+\alpha_{i2}\alpha_{j2}+\mu\alpha_{i3}\alpha_{j3}=0,\qquad (i,j=1,2,3;\ i<j);\end{array}\right\} \tag{121}$$

$$\alpha_{i1}\gamma_{j1}-\alpha_{j1}\gamma_{i1}+\alpha_{i2}\gamma_{j2}-\alpha_{j2}\gamma_{i2}+\alpha_{i3}\gamma_{j3}-\alpha_{j3}\gamma_{i3}=0,\qquad (i,j=1,2,3;\ i<j). \tag{122}$$

Now, if $p > 2$, conditions (121) are precisely the conditions* for the invariance of $\xi_1^2+\xi_2^2+\frac{1}{\mu}\xi_3^2$ under the ternary transformation (α_{ij}), so that the number of sets of solutions is $2(p^{2n}-1)p^n$. For $p=2$, then $\mu=1$ and the first conditions (121) give $\alpha_{i3}=1+\alpha_{i1}+\alpha_{i2}$ for $i=1,2,3$. Then the second set of three become

$$A_{11}A_{22}+A_{12}A_{21}=1,\quad A_{11}A_{32}+A_{12}A_{31}=1,\quad A_{22}A_{31}+A_{21}A_{32}=1\ ,$$

where $A_{ij}\equiv 1+\alpha_{ij}$. The first two of these may be solved for A_{11} and A_{12}, since their determinant is unity by the third:

$$A_{11}=A_{21}+A_{31},\quad A_{12}=A_{22}+A_{32}.$$

The third has $2^n(2^{2n}-1)$ sets of solutions. This is therefore the number of sets of solutions α_{ij} of (121). The determinant of (13) equals $|\alpha_{ij}|^2$. Hence $|\alpha_{ij}|=\pm 1$, as is evident from what precedes if $p>2$. Now if all the determinants of order 3 in the 3 by 9 matrix of the coefficients of the γ_{ij} in (122) all vanish, we find that the minor of each α_{i3} in $|\alpha_{ij}|$ vanishes, whereas the latter

* American Journal, Vol. XXI (1899), p. 195, formulae (1′), (2′) for $s=m=3$.

$\neq 0$. Hence relations (122) determine three of the γ_{ij} in terms of the remaining ones and the α_{ij}. The number of transformations commutative with $L_{1,1}L_{2,1}L_{3,\mu}$ is therefore

$$2(p^{2n}-1)p^n.\,p^{6n} \text{ or } (2^{2n}-1)\,2^n.\,2^{6n}.$$

Types $L_{1,\mu}T$, $L_{1,\mu}L_{2,1}T$. The conditions for $L_{1,\mu}L_{2,\tau}(13)=(13)L_{1,\mu}L_{2,\tau}$ are

$$\beta_{11}=\beta_{12}=\beta_{21}=\beta_{31}=\beta_{13}=\delta_{13}=\alpha_{31}=0,\quad \tau\beta_{22}=0,\quad \tau\beta_{32}=0,$$
$$\tau\beta_{23}=0,\quad \tau\alpha_{32}=0,$$
$$\tau\delta_{23}=0,\quad \delta_{11}=\alpha_{11},\quad \tau\delta_{21}=\mu\alpha_{21},\quad \mu\delta_{12}=\tau\alpha_{12},\quad \tau\delta_{22}=\tau\alpha_{22}.$$

According as $\tau=0$ or $\tau=1$, we obtain

$$\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} & \alpha_{13} & \gamma_{13} \\ 0 & \alpha_{11} & 0 & 0 & 0 & 0 \\ 0 & \gamma_{21} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ 0 & \delta_{21} & \beta_{22} & \delta_{22} & \beta_{23} & \delta_{23} \\ 0 & \gamma_{31} & \alpha_{32} & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ 0 & \delta_{31} & \beta_{32} & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix},\quad \begin{pmatrix} \alpha_{11} & \gamma_{11} & \mu\delta_{12} & \gamma_{12} & \alpha_{13} & \gamma_{13} \\ 0 & \alpha_{11} & 0 & \delta_{12} & 0 & 0 \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ 0 & \mu\alpha_{21} & 0 & \alpha_{22} & 0 & 0 \\ 0 & \gamma_{31} & 0 & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ 0 & \delta_{31} & 0 & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}.$$

For the first, $\alpha_{11}=\pm 1$ and γ_{21}, δ_{21}, γ_{31}, δ_{31} are determined by the Abelian conditions, while the matrix obtained by deleting the first two rows and columns is that of a quaternary special Abelian transformation. The number of transformations commutative with $L_{1,\mu}$ is therefore

$$2p^{5n}.\,p^{4n}(p^{4n}-1)(p^{2n}-1) \text{ or } 2^{5n}.\,2^{4n}(2^{4n}-1)(2^{2n}-1).$$

For the second the Abelian conditions are

$$\alpha_{11}^2+\mu\delta_{12}^2=1,\quad \alpha_{11}\alpha_{21}+\delta_{12}\alpha_{22}=0,\quad \mu\alpha_{21}^2+\alpha_{22}^2=1, \tag{123}$$
$$\alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1,\quad \alpha_{11}\gamma_{21}-\gamma_{11}\alpha_{21}+\mu\delta_{12}\gamma_{22}-\gamma_{12}\alpha_{22}+\alpha_{13}\gamma_{23}-\gamma_{13}\alpha_{23}=0, \tag{124}$$
$$\alpha_{11}\gamma_{31}+\mu\delta_{12}\gamma_{32}+\alpha_{13}\gamma_{33}-\gamma_{13}\alpha_{33}=0,\quad \alpha_{11}\delta_{31}+\mu\delta_{12}\delta_{32}+\alpha_{13}\delta_{33}-\gamma_{13}\beta_{33}=0, \tag{125}$$
$$\alpha_{21}\gamma_{31}+\alpha_{22}\gamma_{32}+\alpha_{23}\gamma_{33}-\gamma_{23}\alpha_{33}=0,\quad \alpha_{21}\delta_{31}+\alpha_{22}\delta_{32}+\alpha_{23}\delta_{33}-\gamma_{23}\beta_{33}=0. \tag{126}$$

From (123), $\alpha_{21}^2=\delta_{12}^2$, $\alpha_{22}=\pm\alpha_{11}$, $\alpha_{11}(\alpha_{21}\pm\delta_{12})=0$. The number N of sets of solutions of (123) is

$$2(p^n+\epsilon) \text{ if } \mu=\nu;\ 2(p^n-\epsilon) \text{ if } \mu=1,\ p>2;\ 2^n \text{ if } p=2$$

where $\epsilon=\pm 1$ according as $p^n=4l\pm 1$.

In view of (124), we can solve (125) for α_{13} and γ_{13}, and (126) for α_{23} and γ_{23}. Conceive of these values being substituted in $(124)_2$. The coefficients of γ_{21} and γ_{11} remain α_{11} and $-\alpha_{21}$ respectively, not both of which are zero. Hence either γ_{11} or γ_{21} is determined. Hence for each of the $p^n(p^{2n}-1)N$ sets of solutions of (123) and $(124)_1$, for the 8 coefficients entering, the 12 remaining coefficients are connected by relations which determine 5 of them. Hence $L_{1,\mu}L_{2,1}$ is commutative with exactly

$$2(p^n+\varepsilon)p^n(p^{2n}-1)p^{7n} \text{ if } \mu=\nu,\ p>2;$$

$$2(p^n-\varepsilon)p^n(p^{2n}-1)p^{7n} \text{ if } \mu=1,\ p>2$$

$$2^n.\,2^n(2^{2n}-1)\,2^{7n} \text{ if } p=2.$$

Type $L_{1,\mu}T_{1,-1}T$, $p>2$. A transformation commutative with it is commutative with its p^{th} power $T_{1,-1}T$ and hence is the product of a special Abelian transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on ξ_1, η_1 by one on ξ_2, η_2, ξ_3, η_3. Being commutative with $L_{1,\mu}$, $a=d=\pm 1$, $c=0$. Hence there are

$$2p^n.\,p^{4n}(p^{4n}-1)(p^{2n}-1).$$

Type $L_{2,\mu}T_{1,-1}T$, $p>2$. A substitution commutative with it must be commutative with both $T_{1,-1}$ and $L_{2,\mu}$. By Trans., bottom of page 114, or by type $L_{1,\mu}$, their number is

$$2p^{4n}(p^{2n}-1)\,.\,p^n(p^{2n}-1).$$

Type $L_{1,\mu}T_{1,-1}L_{2,\lambda}T$, $p>2$, μ, $\lambda=1$, ν. A transformation commutative with it is commutative with $T_{1,-1}$ and $L_{1,\mu}L_{2,\lambda}$. The number is thus

$$2p^{4n}(p^{2n}-1)\,.\,2p^n.$$

Type $L_{2,\mu}T_{2,-1}L_{3,1}T_{3,-1}T$, $p>2$. Its p^{th} power is $T_{2,-1}T_{3,-1}T$. A commutative transformation is, therefore, the product of type (17), Trans., p. 115, on ξ_2, η_2, ξ_3, η_3 by a transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant unity on ξ_1, η_1. The number is thus

$$2p^{3n}(p^n+\varepsilon)\,.\,p^n(p^{2n}-1) \text{ if } \mu=\nu;\quad 2p^{3n}(p^n-\varepsilon)p^n(p^{2n}-1) \text{ if } \mu=1.$$

Type $L_{1,\lambda}T_{1,-1}L_{2,\mu}L_{3,1}T$, $p>2$. A commutative transformation is of the form in the preceding type with $c=0$, $a=d=\pm 1$:

$$2p^{3n}(p^n+\varepsilon)\,.\,2p^n \text{ if } \mu=\nu\,;\quad 2p^{3n}(p^n-\varepsilon)\,.\,2p^n \text{ if } \mu=1\,.$$

Type $T_{3,-1}T$, $p>2$. The number of commutative ones is

$$p^n(p^{2n}-1)\,.\,p^{4n}(p^{4n}-1)(p^{2n}-1).$$

Type $R_{1,2,1}L'_{3,1}$, $p=2$. For the sequence $\xi_1, \eta_2, \xi_2, \eta_1, \xi_3, \eta_3$ of the variables, the commutative transformation (as given by inspection by the general theory) is the first one below. For the initial sequence $\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3$, we have the second:

$$\begin{pmatrix} a & 0 & e & 0 & \alpha & 0 \\ b & a & f & e & \beta & \alpha \\ g & 0 & c & 0 & \gamma & 0 \\ h & g & d & c & \delta & \gamma \\ \varkappa & 0 & \rho & 0 & r & 0 \\ \tau & \varkappa & \sigma & \rho & s & r \end{pmatrix}, \quad \begin{pmatrix} a & 0 & e & 0 & \alpha & 0 \\ h & c & d & g & \delta & \gamma \\ g & 0 & c & 0 & \gamma & 0 \\ b & e & f & a & \beta & \alpha \\ \varkappa & 0 & \rho & 0 & r & 0 \\ \tau & \rho & \sigma & \varkappa & s & r \end{pmatrix}.$$

The Abelian conditions on the second give

$$\alpha=0,\quad \gamma=0,\quad r=1,\quad a\rho+e\varkappa=0,\quad c\varkappa+g\rho=0, \tag{127}$$

$$he+bc+da+fg=0,\quad h\rho+c\tau+d\varkappa+g\sigma+\delta r=0, \tag{128}$$

$$ac+eg=1,\quad b\rho+e\tau+f\varkappa+a\sigma+\beta r=0. \tag{129}$$

From the last two in (127), $\rho=\varkappa=0$, since the determinant is not zero by $(129)_1$. Now $(128)_2$ and $(129)_2$ determine δ and β. Since e and c are not both zero, $(128)_1$ determines either h or b. Hence of the 18 coefficients, 8 are uniquely determined in terms of a, c, e, g [connected by $(129)_1$], d, f, τ, σ, s and either h or b. The number of transformations is therefore

$$2^n(2^{2n}-1)\,2^{6n}.$$

This result suggests that the type is perhaps conjugate with $L_{1,1}L_{2,1}L_{3,1}$. That this is in fact the case follows since the Abelian transformation

$$\xi'_1=\eta_2+\eta_3,\quad \eta'_1=\xi_1+\xi_3,\quad \xi'_2=\eta_1+\eta_3,$$
$$\eta'_2=\xi_2+\xi_3,\quad \xi'_3=\xi_1+\xi_2+\xi_3,\quad \eta'_3=\eta_1+\eta_2+\eta_3$$

transforms $L_{1,1}L_{2,1}L_{3,1}$ into $R_{1,2,1}L_{3,1}$.

Type $R_{1,2,1}$ for $p=2$. The auxiliary and final are now

$$\begin{pmatrix} a & 0 & e & 0 & 0 & 0 \\ b & a & f & e & \alpha & \gamma \\ g & 0 & c & 0 & 0 & 0 \\ h & g & d & c & \beta & \delta \\ x & 0 & \rho & 0 & r & s \\ \tau & 0 & \sigma & 0 & t & u \end{pmatrix}, \quad \begin{pmatrix} a & 0 & e & 0 & 0 & 0 \\ h & c & d & g & \beta & \delta \\ g & 0 & c & 0 & 0 & 0 \\ b & e & f & a & \alpha & \gamma \\ x & 0 & \rho & 0 & r & s \\ \tau & 0 & \sigma & 0 & t & u \end{pmatrix}.$$

The Abelian conditions on the second are

$$ac+eg=1, \quad ex+a\rho+\alpha s+\gamma r=0, \quad e\tau+a\sigma+\alpha u+\gamma t=0, \tag{130}$$
$$ru+ts=1, \quad he+bc+da+fg+\beta\gamma+\alpha\delta=0, \tag{131}$$
$$cx+g\rho+\beta s+r\delta=0, \quad c\tau+g\sigma+\beta u+t\delta=0. \tag{132}$$

By the last two in (130), α and γ are determined, since the determinant $\neq 0$ by $(131)_1$. Likewise β and δ are determined by (132). Then $(131)_2$ determines either h or b, neither occurring in the other conditions. There are $[2^n(2^{2n}-1)]^2$ sets, a, c, e, g, r, u, t, s, satisfying $(130)_1$, $(131)_1$. Then $d, f, x, \rho, \tau, \sigma$, and either h or b, are arbitrary. The number of transformations is thus $2^{9n}(2^{2n}-1)^2$.

Type $R_\beta L'_{3,\tau}$, $p=2$. Denote it by $V_{\beta,\tau}$. If S be defined by (13), we have $V_{\beta,\tau}S=SV_{B,T}$ if, and only if,

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \delta_{12} & 0 & \alpha_{11} & 0 & 0 & 0 \\ \beta_{21} & 0 & \delta_{12} & \alpha_{11} & \tau\delta_{13} & 0 \\ \alpha_{31} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \\ \beta_{31} & 0 & T\alpha_{31} & 0 & \beta_{33} & \delta_{33} \end{pmatrix},$$

$$\tau\gamma_{33}=T\gamma_{33}=0, \quad \tau\delta_{33}=T\alpha_{33}, \quad \beta\alpha_{11}+\beta_{12}+\delta_{12}=B\alpha_{11}+\beta_{21}.$$

The Abelian conditions all reduce to

$$\alpha_{11}=1, \quad \beta_{21}+\delta_{12}^2+\beta_{12}+\tau\delta_{13}^2=0, \quad \beta_{31}+T\alpha_{31}\delta_{12}+\beta_{13}\delta_{33}+\delta_{13}\beta_{33}=0,$$
$$\alpha_{31}+\beta_{13}\gamma_{33}+\delta_{13}\alpha_{33}=0, \quad T\alpha_{31}+\tau\delta_{13}\delta_{33}=0, \quad \alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1.$$

If $\tau=1$, then $\gamma_{33}=0$, $\delta_{33}=T\alpha_{33}$, $\alpha_{33}\delta_{33}=1$, whence $T\neq 0$. By an earlier transformation, we can make $T=1$. If $\tau=0$, then $T\gamma_{33}=T\alpha_{33}=0$, whence $T=0$. Hence, if the two are conjugate, we may set $T=\tau=1$ or 0.

For $T=\tau=1$, then $\gamma_{33}=0$, $\alpha_{33}=\delta_{33}=1$, $\alpha_{11}=1$, $\alpha_{31}=\delta_{13}$,

$$\beta_{31}=\alpha_{31}\delta_{12}+\beta_{13}+\delta_{13}\beta_{33},\quad \beta_{21}=\beta_{12}+\delta_{12}^2+\delta_{13}^2,\quad \delta_{13}^2=\delta_{12}+\delta_{12}^2+\beta+B.$$

Hence $V_{\beta,1}$ is transformed into $V_{B,1}$ by the resulting Abelian transformation in which δ_{12}, β_{12}, β_{33}, β_{13}, β_{11} are arbitrary* and the others determined. It follows that the number commutative with $V_{\beta,1}$ is 2^{5n}.

For $T=\tau=0$, we must have $\delta_{12}^2+\delta_{12}=\beta+B$, so that a necessary condition is $f(\beta)=f(B)$, f defined by (28). Let this condition be satisfied, so that two values of δ_{12} can be found in the $GF[2^n]$. Then for any set of solutions of $\alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1$ and β_{11}, β_{12}, β_{13}, δ_{13} arbitrary, the conditions merely determine β_{11}, β_{21}, α_{31}, β_{31}. Hence there are exactly two non-conjugate forms $V_{0,0}$ and $V_{b,0}$, b a particular solution of $f(\eta)=1$. It follows also that there are exactly $2\,.\,2^n(2^{2n}-1)\,2_{4n}$ Abelian transformations commutative with R_β, viz.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \delta_{12} & 0 & 1 & 0 & 0 & 0 \\ \beta_{12}+\delta_{12} & 0 & \delta_{12} & 1 & 0 & 0 \\ \beta_{13}\gamma_{33}+\delta_{13}\alpha_{33} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \\ \beta_{13}\delta_{33}+\delta_{13}'\beta_{33} & 0 & 0 & 0 & \beta_{33} & \delta_{33} \end{pmatrix},\qquad \begin{array}{l} \delta_{12}=0 \text{ or } 1, \\ \alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1. \end{array}$$

Types $A_\mu T_{1,\pm1}T_{2,\pm1}T_{3,c}L'_{3,\tau}$, $c^2=1$, $\tau=0,1,\nu$, $p>2$. It suffices to consider $A_\mu L'_{3,\tau}$, $A_\mu T_{3,-1}L'_{3,\tau}$ and their products by $T_{1,-1}T_{2,-1}T_{3,-1}$.

Type $A_\mu T_{3,-1}L'_{3,\tau}T\equiv Y_{\mu,\tau}$, $p>2$. The power p^2 of it gives $T_{3,-1}T$, since A_μ is of period p if $p>3$ and of period 9 if $p=3$ (Trans., p. 113). Hence if S transforms $Y_{\mu,\tau}$ into $Y_{M,t}$, S is commutative with $T_{3,-1}$ and hence is the product of a special Abelian transformation Y on ξ_1, η_1, ξ_2, η_2 by one on ξ_3, η_3. The latter, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, must transform $L'_{3,\tau}$ into $L'_{3,t}$. Hence

$$b\tau=0,\quad bt=0,\quad d\tau=at,\quad ad-bc=1.$$

If $\tau=0$, then $t=0$. If $\tau\neq0$, then $ad=1$, $d\tau=at$, whence t/τ must be a square. Applying earlier transformations, we may take $\tau=t=1$ or ν. Since Y shall transform A_μ into A_M, where μ, $M=1$, ν, it follows from §9 that $\mu=M$. Hence there are 6 non-conjugate forms $Y_{\mu,\tau}$ ($\mu=1,\nu$; $\tau=0,1,\nu$). We may

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* Taking these zero, whence $\beta_{31}=0$, $\beta_{21}=\delta_{13}^2$, $\delta_{13}^2=\beta+B$, I verified the conjugacy.

now readily determine the transformations S commutative with each $Y_{\mu,\tau}$. By §9, for $\alpha=\alpha'=\mu$, we have

$$Y=\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \delta_{12} \\ -\delta_{12} & 0 & \alpha_{11} & 0 \\ -\beta_{12}-\mu\delta_{12} & 0 & \mu\delta_{12} & \alpha_{11} \end{pmatrix}, \quad \begin{array}{l} \alpha_{11}^2=1, \\ 2\alpha_{11}\beta_{12}+\mu\alpha_{11}\delta_{12}-\mu\delta_{12}^2=0, \end{array}$$

the number of which is $2p^{2n}$. Further, we have

$$b\tau=0, \quad d\tau=a\tau, \quad ad-bc=1,$$

the number of solutions of which is $p^n(p^{2n}-1)$ or $2p^n$ according as $\tau=0$ or $\tau\neq 0$. In the respective cases, $Y_{\mu,\tau}$ is commutative with

$$2p^{3n}(p^{2n}-1) \text{ or } 4p^{3n}.$$

Type $A_\mu L'_{3,\tau}T\equiv X_{\mu,\tau}$, $\mu=1,\nu$; $\tau=0,1,\nu$, $p>2$. The conditions for $X_{\mu,\tau}S=SX_{M,t}$ require that

$$S=\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{11} & 0 & 0 & 0 \\ -\beta_{12}-\mu\delta_{12} & 0 & \mu\delta_{12} & \delta_{11} & \tau\delta_{13} & 0 \\ \alpha_{31} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \\ \beta_{31} & 0 & -t\alpha_{31} & 0 & \beta_{33} & \delta_{33} \end{pmatrix}, \quad \begin{array}{rl} \mu\delta_{11}= & M\alpha_{11}, \\ \mu\delta_{12}= & -M\alpha_{21}, \\ \tau\gamma_{33}= & 0, \\ t\gamma_{33}= & 0, \\ \tau\delta_{33}= & t\alpha_{33}. \end{array}$$

The Abelian conditions all give

$$\begin{gathered} \alpha_{11}\delta_{11}=1, \quad \delta_{11}\alpha_{21}+\alpha_{11}\delta_{12}=0, \quad 2\delta_{11}\beta_{12}+\mu\delta_{11}\delta_{12}-\mu\delta_{12}^2-\tau\delta_{13}^2=0, \\ t\delta_{11}\alpha_{31}+\tau\delta_{13}\delta_{33}=0, \quad -\delta_{11}\beta_{31}+t\delta_{12}\alpha_{31}+\beta_{13}\delta_{33}-\delta_{13}\beta_{33}=0, \\ \alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1, \quad -\delta_{11}\alpha_{31}+\beta_{13}\gamma_{33}-\delta_{13}\alpha_{33}=0. \end{gathered}$$

Then $\mu\delta_{11}^2=M$, whence $M=\mu=1$ or ν; $\delta_{11}=\alpha_{11}=\pm 1$, $\alpha_{21}=-\delta_{12}$. If $\tau=0$, then $t=0$. If $\tau\neq 0$, then $\gamma_{33}=0$, $\alpha_{33}\delta_{33}=1$, $t=\tau\delta_{33}^2$, whence $t=\tau$. Hence no two of the six forms $X_{\mu,\tau}$ are conjugate.

Setting $M=\mu$, $t=\tau$, we get $\delta_{11}=\alpha_{11}=\pm 1$, $\alpha_{21}=-\delta_{12}$ and

$$\tau\gamma_{33}=0, \quad \tau\delta_{33}=\tau\alpha_{33}, \quad \pm 2\beta_{12}\pm\mu\delta_{12}-\mu\delta_{12}^2-\tau\delta_{13}^2=0, \tag{133}$$

$$\pm\tau\alpha_{31}+\tau\delta_{13}\delta_{33}=0, \quad \pm\beta_{31}=\tau\delta_{12}\alpha_{31}+\beta_{13}\delta_{33}-\delta_{13}\beta_{33}, \tag{134}$$

$$\alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1, \quad \pm\alpha_{31}=\beta_{13}\gamma_{33}-\delta_{13}\alpha_{33}. \tag{135}$$

If $\tau\neq 0$, then $\gamma_{33}=0$, $\delta_{33}=\alpha_{33}$, $\alpha_{33}^2=1$, $\alpha_{31}=\mp\delta_{13}\alpha_{33}$, while the last conditions in (133) and (134) determine β_{12} and β_{31}. Since β_{11}, δ_{12}, β_{13}, δ_{13}, β_{33} are arbitrary,

we obtain $4p^{5n}$ transformations commutative with $X_{\mu,\tau}$, $\tau \neq 0$. If $\tau = 0$, β_{11}, δ_{12}, β_{13} and δ_{13} are arbitrary, while α_{11}, δ_{11}, α_{21}, β_{12}, β_{31}, α_{31} are determined. Hence there are $2p^{4n}.p^n(p^{2n}-1)$ transformations commutative with A_μ.

It remains to take as W, in $WT_{3,\pm 1}L_{3,\tau}$, a quaternary canonical form for which $D(\rho)$ has four roots not in the $GF[p^n]$, or two roots ± 1 and two roots not in the $GF[p^n]$. Consider the subdivisions necessary for the types $T_{1,c}L_{1,b}T_{2,\lambda}T_{3,\pm 1}L_{3,\tau}$, where (to the end of this section) $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$. A transformation which transforms one such type into another is the product of $T_{2,\delta}$, $\delta^{p^n+1} = 1$, by a special Abelian transformation A on $\xi_1, \eta_1, \xi_3, \eta_3$. Hence they must have the same c and the same sign $\pm$ in view of the invariance of the roots of their characteristic equations. If $c = \pm 1$, we may restrict $L_{1,b}L_{3,\tau}$, to the forms I, $L_{1,\mu}$, $L_{1,1}L_{3,\mu}$. If $c = \mp 1$, $p > 2$, it suffices to take as $T_{1,c}L_{1,b}T_{3,\pm 1}L_{3,\tau}$ one of the non-conjugate forms $T_{1,-1}$, $T_{1,-1}L_{1,\mu}$, $T_{1,-1}L_{3,\mu}$, $T_{1,-1}L_{1,\mu}L_{3,\tau}$, where $\tau = 1$ or ν.

Type $T_{1,\pm 1}T_{2,\lambda}T_{3,\pm 1}$. The number of non-conjugate forms is

$$p^n - 1 \text{ or } \tfrac{1}{2}2^n.$$

The number of Abelian transformations C commutative with any one of the forms is

$$p^{4n}(p^{4n}-1)(p^{2n}-1)(p^n+1).$$

Type $T_{1,\pm 1}L_{1,\mu}T_{2,\lambda}T_{3,\pm 1}$. The number of non-conjugates is

$$2(p^n-1) \text{ or } \tfrac{1}{2}2^n.$$

The number of the C is (Trans., bottom of p. 114)

$$2p^{4n}(p^{2n}-1)(p^n+1) \text{ or } 2^{4n}(2^{2n}-1)(2^n+1).$$

Type $T_{1,\pm 1}L_{1,1}T_{2,\lambda}T_{3,\pm 1}L_{3,\mu}$. There are

$$2(p^n_{\mu}-1) \text{ or } \tfrac{1}{2}2^n$$

non-conjugate forms. The number of the C is (Tr., p. 115)

$2p^{3n}(p^n+\epsilon)(p^n+1)$ if $\mu = \nu$, $p > 2$;

$$2p^{3n}(p^n-\epsilon)(p^n+1) \text{ if } \mu = 1,\ p > 2;\ 2^{4n}(2^n+1) \text{ if } p = 2.$$

Type $T_{1,-1}T_{2,\lambda}$, $p > 2$. There are $\frac{1}{2}(p^n-1)$ non-conjugate forms, each commutative with $p^{2n}(p^{2n}-1)^2(p^n+1)$ transformations.

Types $T_{1,-1}L_{1,\mu}T_{2,\lambda}$, $T_{1,-1}T_{2,\lambda}L_{3,\mu}$, $p > 2$. The total number of non-conju-

gates is $2(p^n-1)$. Each is commutative (Tr., §9) with

$$2p^{3n}(p^{2n}-1)(p^n+1).$$

Type $T_{1,-1}L_{1,\mu}T_{2,\lambda}L_{3,\tau}$, $p>2$. Each of the $2(p^n-1)$ non-conjugate forms is commutative (Tr., §10) with $4p^{2n}(p^n+1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}L_{3,\mu}$, $\sigma^{p^{3n}+1}=1$, $\sigma^3\neq 1$. Each of the

$$p^{2n}-1 \text{ or } \tfrac{1}{4}\,2^{2n}$$

non-conjugate forms is commutative with $2p^n(p^{3n}+1)$ or $2^n(2^{2n}+1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}$, σ as before. Each of the

$$\tfrac{1}{2}(p^{2n}-1) \text{ or } \tfrac{1}{4}\,2^{2n}$$

non-conjugate forms is commutative with $p^n(p^{2n}-1)(p^{3n}+1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}L_{3,\mu}$, $\sigma^{p^{2n}-1}=1$, $\sigma^{p^n\pm 1}\neq 1$. Each of the

$$(p^n-1)^2 \text{ or } \tfrac{1}{4}\,2^n(2^n-2)$$

non-conjugate forms is commutative with $2p^n(p^{2n}-1)$ or $2^n(2^{2n}-1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}$, σ as in preceding. Each of the

$$\tfrac{1}{2}(p^n-1)^2 \text{ or } \tfrac{1}{4}\,2^n(2^n-2)$$

non-conjugate forms is commutative with $p^n(p^{2n}-1)^2$.

Type $T_{1,\varkappa}T_{2,\lambda}T_{3,\pm 1}L_{3,\mu}$, $\varkappa^{p^n+1}=1$, $\lambda^{p^n+1}=1$, λ, λ^{-1}, $\varkappa$, $\varkappa^{-1}$ all distinct. The number of non-conjugate forms is

$$\tfrac{1}{2}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{8}\,2^n(2^n-2).$$

Each is commutative with $2p^n(p^n+1)^2$ or $2^n(2^n+1)^2$.

Type $T_{1,\varkappa}T_{2,\lambda}T_{3,\pm 1}$, $\varkappa$, λ as before. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{8}\,2^n(2^n-2)$$

non-conjugate forms each commutative with $p^n(p^{2n}-1)(p^n+1)^2$.

Type $T_{1,\varkappa}T_{2,\varkappa}T_{3,\pm 1}L_{3,\mu}$, $\varkappa^{p^n+1}=1$, $\varkappa^2\neq 1$. There are

$$2(p^n-1) \text{ or } \tfrac{1}{2}\,2^n$$

non-conjugate forms, each commutative with (Tr., §21)

$$2p^n.(p^n+1)p^n(p^{2n}-1) \text{ or } 2^n.(2^n+1)\,2^n(2^{2n}-1).$$

Type $T_{1,\varkappa}T_{2,\varkappa}T_{3,\pm 1}$, $\varkappa$ as before. There are p^n-1 or $\frac{1}{2}2^n$ non-conjugate forms, each commutative with

$$p^n(p^{2n}-1).(p^n+1)p^n(p^{2n}-1).$$

Type (63) $T_{3,\pm 1}L_{3,\mu}$, $x^{p^n+1}=1$, $x^2 \neq 1$. There are

$$2(p^n-1) \text{ or } \tfrac{1}{2}\, 2^n$$

non-conjugate forms, each commutative with (Tr., §22)

$$2p^n \cdot p^n(p^n+1) \text{ or } 2^n \cdot 2^n(2^n+1).$$

Type (63) $T_{3,\pm 1}$, x as before. Each of the p^n-1 or $\frac{1}{2}\,2^n$ non-conjugate forms is commutative with $p^n(p^n+1)\,.\,p^n(p^{2n}-1)$.

40. Finally, consider transformation (55) for $m=3$ and F the $GF[p^n]$. The hypothesis is that its characteristic equation has no root in F. Changing the notation and applying §1, we may write it $V\,T_{3,\lambda}$, where V is a special Abelian transformation on $\xi_1, \eta_1, \xi_2, \eta_2$, and where, as always in this section, $\lambda^{p^n+1}=1$, $\lambda^2 \neq 1$. Likewise for $\varkappa$ and τ.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\lambda}$, $\sigma^{p^{2n}+1}=1$, $\sigma^2 \neq 1$. Each of the

$$\tfrac{1}{8}(p^{2n}-1)(p^n-1) \text{ or } \tfrac{1}{8}\, 2^{3n}$$

non-conjugate forms is commutative with $(p^{2n}+1)(p^n+1)$ transformations.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\lambda}$, $\sigma^{p^{2n}-1}=1$, $\sigma^{p^n\pm 1} \neq 1$. Each of the

$$\tfrac{1}{8}(p^n-1)^3 \text{ or } \tfrac{1}{8}\, 2^{2n}(2^n-2)$$

non-conjugate forms is commutative with $(p^{2n}-1)(p^n+1)$.

Type $T_{1,\varkappa}T_{2,\tau}T_{3,\lambda}$, $\varkappa, \varkappa^{-1}, \tau, \tau^{-1}, \lambda, \lambda^{-1}$ all distinct roots of $x^{p^n+1}=1$. The number of non-conjugate forms is

$$\tfrac{1}{48}(p^n-1)(p^n-3)(p^n-5) \text{ or } \tfrac{1}{48}\, 2^n(2^n-2)(2^n-4).$$

Each is commutative with $(p^n+1)^3$ transformations.

Type $T_{1,\varkappa}T_{2,\varkappa}T_{3,\lambda}$, $\varkappa, \varkappa^{-1}, \lambda, \lambda^{-1}$ all distinct. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4}\, 2^n(2^n-2)$$

non-conjugate forms, each commutative with (Tr., §21)

$$(p^n+1)\,p^n(p^{2n}-1)\,.\,(p^n+1).$$

Type $T_{1,\lambda}T_{2,\lambda}T_{3,\lambda}$. There are $\frac{1}{2}(p^n-1)$ or $\frac{1}{2}\,2^n$ non-conjugate forms. By the general theory each is commutative with only S:

$$S=\begin{pmatrix} a & 0 & b & 0 & c & 0 \\ 0 & a^{p^n} & 0 & b^{p^n} & 0 & c^{p^n} \\ e & 0 & f & 0 & g & 0 \\ 0 & e^{p^n} & 0 & f^{p^n} & 0 & g^{p^n} \\ h & 0 & j & 0 & l & 0 \\ 0 & h^{p^n} & 0 & j^{p^n} & 0 & l^{p^n} \end{pmatrix}, \quad \Sigma=\begin{pmatrix} a & b & c \\ e & f & g \\ h & j & l \end{pmatrix}.$$

The 6 conditions that S shall be Abelian are precisely the conditions that Σ shall be hyperorthogonal (Linear Groups, p. 133), the number of which is

$$(p^{3n}+1)p^{2n}(p^{2n}-1)p^n(p^n+1).$$

Type (63) $T_{3,\lambda}$, $\lambda \neq \kappa, \kappa^{-1}$. The number of non-conjugate forms is

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4}2^n(2^n-2).$$

Each is commutative with $p^n(p^n+1)^2$ transformations.

Type (63) $T_{3,\kappa}$. There are $\frac{1}{2}(p^n-1)$ or $\frac{1}{2}2^n$ non-conjugate forms. Each is commutative with only (118), where now

$$a_1=a^{p^n},\quad b_1=b^{p^n},\quad c_1=c^{p^n},\quad d_1=d^{p^n},\quad e_1=e^{p^n}.$$

The resulting transformations are, however, not Abelian, the present case being unique in this respect. Indeed, by §19, end, a transformation of $SA(4, p^n)$ is reducible to its canonical form (63) by a special Abelian transformation in the $GF[p^{2n}]$. It is otherwise with $T_{3,\kappa}$, $\kappa^{p^n+1}=1$. It follows from §1 that we may assume as the transformation of variables to obtain $T_{3,\kappa}$

$$x_3=(\xi_3-\gamma\kappa\eta_3)/f(\kappa),\quad y_3=(\xi_3-\gamma\kappa^{-1}\eta_3)/f(\kappa^{-1}),$$

$f(\kappa)$ being linear in κ with coefficients in the $GF[p^n]$. We can make

$$f(\kappa)\cdot f(\kappa^{-1})=\gamma(\kappa-\kappa^{-1})^2,$$

by choice of the coefficients in $f(\kappa)$. For our transformation of variables,

$$\sum_{i=1}^{3}\begin{vmatrix}\xi_i & \eta_i\\ \xi_i^* & \eta_i^*\end{vmatrix}=\begin{vmatrix}x_1 & y_1\\ x_1^* & y_1^*\end{vmatrix}+\begin{vmatrix}x_2 & y_2\\ x_2^* & y_2^*\end{vmatrix}+(\kappa-\kappa^{-1})\begin{vmatrix}x_3 & y_3\\ x_3^* & y_3^*\end{vmatrix}.$$

Transformation (118) must leave the second member unaltered. Hence*

$$a^{p^n+1}=1,\quad e^{p^n+1}=1,\quad ab^{p^n}-a^{p^n}b+(\kappa-\kappa^{-1})d^{p^n+1}=0,$$
$$ac^{p^n}+(\kappa-\kappa^{-1})de^{p^n}=0,\quad a^{p^n}c-(\kappa-\kappa^{-1})d^{p^n}e=0.$$

One of the last two may be dropped. The other determines c. Now

$$d^{p^n+1}=\left[\left(\frac{b}{a}\right)^{p^n}-\frac{b}{a}\right]\Big|(\kappa^{-1}-\kappa).$$

For any b/a in the $GF[p^{2n}]$, the second member equals its conjugate, and hence belongs to the $GF[p^n]$. Hence if b/a is not in the $GF[p^n]$, there are exactly p^n+1 roots d in the $GF[p^{2n}]$; otherwise, $d=0$. The number of transforma-

* Modifying §2 for $m=3$, we define C'_{rs} from C_{rs} by inserting the factor $\kappa-\kappa^{-1}$ before the term given by $l=3$ in (3). Then $C'_{12}=C'_{34}=1$, $C'_{56}=\kappa-\kappa^{-1}$; all remaining $C'_{rs}=0$.

tions of $SA(6, p^n)$ commutative with one having the canonical form (63) $T_{3,\kappa}$ is therefore *

$$(p^n+1)^2\,[p^n+(p^{2n}-p^n)(p^n+1)]\equiv p^{3n}(p^n+1)^2.$$

The remaining types in §§7–33 *for* $m=3$.

41. We now discuss for $m=3$ the canonical forms determined in §§7–33, aside from types (16), (20), (23) and (55), which have been exhaustively treated in §§34–40.

Type (22). There are $\frac{1}{2}(p^n-3)$ or $\frac{1}{2}(2^n-2)$ non-conjugate forms. Indeed, $P_{12}M_3^{-1}T_{2,-1}Q_{3,1,1}$ transforms (22) into a transformation derived from (22) by replacing x by x^{-1} throughout. The only linear transformations (13) commutative with (22) are

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_{11} & \beta_{12} & 0 & 0 & \delta_{13} \\ 0 & 0 & \delta_{11} & 0 & 0 & 0 \\ \beta_{21} & 0 & 0 & \alpha_{11} & \beta_{23} & 0 \\ -\beta_{23} & 0 & 0 & 0 & \alpha_{11} & 0 \\ 0 & 0 & \delta_{13} & 0 & 0 & \delta_{11} \end{pmatrix}.$$

The Abelian conditions reduce to

$$\delta_{11}=\alpha_{11}^{-1},\quad \beta_{21}=\alpha_{11}^2\beta_{12}-\alpha_{11}^3\delta_{13}^2,\quad \beta_{23}=\alpha_{11}^2\delta_{13}.$$

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* I give a new proof for $p^n=2^n$, and a check for $p>2$. In each case *the work is carried out in the initial field.*

Let first p^n be of the form $4l+3$, so that -1 is a not-square. Then the special Abelian transformation $\Sigma\equiv P_{12}L_{1,-1}T_{1,-1}M_3$ has the canonical form (63) $T_{3,\kappa}$, $\kappa^2=-1$. If (13) be commutative with Σ, it must be commutative with $\Sigma^4=L_{1,1}L_{2,1}$ and hence have the form of the second matrix just preceding (123), for $\mu=1$. The latter is commutative with Σ if, and only if,

$$\alpha_{21}=-\delta_{12},\quad \alpha_{22}=\alpha_{11},\quad \gamma_{22}=\gamma_{11},\quad \gamma_{32}=\delta_{31},\quad \delta_{32}=-\gamma_{31},$$
$$\alpha_{23}=\gamma_{13},\quad \gamma_{23}=-\alpha_{13},\quad \beta_{33}=-\gamma_{33},\quad \delta_{33}=\alpha_{33},\quad \gamma_{12}=\alpha_{21}-\gamma_{21}.$$

The Abelian conditions then reduce to

$$\alpha_{11}^2+\delta_{12}^2=1,\quad \alpha_{33}^2+\gamma_{33}^2=1,\quad \alpha_{11}\delta_{12}+2\alpha_{11}\gamma_{12}-2\gamma_{11}\delta_{12}+\alpha_{13}^2+\gamma_{13}^2=0,$$
$$\alpha_{11}\gamma_{31}+\delta_{12}\delta_{31}+\alpha_{13}\gamma_{33}-\gamma_{13}\alpha_{33}=0,\quad \alpha_{11}\delta_{31}-\delta_{12}\gamma_{31}+\alpha_{13}\alpha_{33}+\gamma_{13}\gamma_{33}=0.$$

The last two determine α_{13} and γ_{13}, the determinant of their coefficients being unity. Since $p>2$, the third determines either γ_{12} or γ_{11}, which do not occur in the other conditions. Now $x^2+y^2=1$ has p^n+1 sets of solutions in the field of order $p^n=4l+3$. Since γ_{31}, δ_{31} and either γ_{12} or γ_{11} remain arbitrary, there are $p^{3n}(p^n+1)^2$ sets of solutions.

For the $GF[2^n]$, a transformation of $SA(6, 2^n)$ having the canonical form (63) $T_{3,\kappa}$ is

$$\xi_1'=\eta_1,\quad \eta_1'=\xi_1+\delta\eta_1+\xi_2,\quad \xi_2'=\eta\ +\delta\xi_2+\eta_2,\quad \eta_2'=\xi_2,\quad \xi_3'=\eta\ ,\quad \eta_3'=\xi_3+(\delta+1)\,\eta_3,$$

The number of transformations commutative with (22) is thus $p^{2n}(p^n - 1)$.

Type $R_\beta T_{3,\lambda}$, $p = 2$, where (as below) $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$. The number of non-conjugate forms is 2^n. Each is commutative (Trans., p. 112) with only $2 \cdot 2^{3n}(2^n + 1)$ Abelian transformations.

Type $R_{1,2,1}T_{3,\lambda}$, $p = 2$. There are $\frac{1}{2}\,2^n$ non-conjugate forms, each commutative with $2^{4n}(2^{2n} - 1)(2^n + 1)$ (Trans., p. 111).

Type $A_\mu T_{1,\pm 1}T_{2,\pm 1}T_{3,\lambda}$, $p > 2$. There are $2(p^n - 1)$ non-conjugate forms, each commutative with $2p^{2n}(p^n + 1)$ (Tr., p. 113).

Type $S_{0,1,-1}T_{1,\pm 1}T_{2,\pm 1}T_{3,\pm 1}$, $p > 2$. An Abelian transformation commutative with either of the two is of the form (36), subject to the conditions (37), (38), R_{12}, R_{24}, R_{34}, R_{23} of §11, where now $b = c = 0$ (it being allowable to substitute R_{23} for R_{46}). Let

$$\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33} = B,\quad \beta_{22} + \beta_{23} = \rho,\quad \beta_{23} + \beta_{33} = \sigma,\quad \beta_{12} + \beta_{13} = \kappa. \tag{136}$$

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whose characteristic determinant is the cube of $\rho^2 - (\delta + 1)\rho + 1$, which is assumed to be irreducible in the field. Forming the conditions that (13) $S = S(13)$, we find that (13) must have the form

$$\begin{pmatrix} a_{11} & \gamma_{11} & a_{12} & \gamma_{12} & a_{13} & \gamma_{13} \\ \gamma_{11} & a_{11} + \delta\gamma_{11} + a_{12} & \gamma_{11} + \delta a_{12} + \gamma_{12} & a_{12} & \gamma_{13} & a_{13} + (\delta + 1)\gamma_{13} \\ a_{12} & \gamma_{11} + \delta a_{12} + \gamma_{12} & a_{11} + \delta\gamma_{11} + a_{12} & \gamma_{11} & \gamma_{13} & a_{13} + (\delta + 1)\gamma_{13} \\ \gamma_{12} & a_{12} & \gamma_{11} & a_{11} & a_{13} & \gamma_{13} \\ a_{31} & \gamma_{31} & \gamma_{31} & a_{31} & a_{33} & \gamma_{33} \\ \gamma_{31} & a_{31} + (\delta + 1)\gamma_{31} & a_{31} + (\delta + 1)\gamma_{31} & \gamma_{31} & \gamma_{33} & a_{33} + (\delta + 1)\gamma_{33} \end{pmatrix}.$$

Consider the Abelian conditions on the latter. We have

$$R_{56} \equiv a_{33}^2 + (\delta + 1)a_{33}\gamma_{33} + \gamma_{33}^2 = 1,\quad R_{14} \equiv 0,\quad R_{23} \equiv 0,$$
$$R_{24} \equiv R_{13} \equiv a_{11}\gamma_{11} + \delta a_{11}a_{12} + \delta\gamma_{11}\gamma_{12} + a_{12}\gamma_{12} + a_{13}^2 + (\delta + 1)a_{13}\gamma_{13} + \gamma_{13}^2 = 0,$$
$$(s)\quad R_{12} + R_{13} \equiv (a_{11} + \gamma_{12})^2 + (\delta + 1)(a_{11} + \gamma_{12})(\gamma_{11} + a_{12}) + (\gamma_{11} + a_{12})^2 = 1,$$
$$R_{16} \equiv a_{11}a_{31} + \gamma_{11}\gamma_{31} + a_{12}\gamma_{31} + \gamma_{12}a_{31} + a_{13}a_{33} + \gamma_{13}\gamma_{33} + (\delta + 1)(a_{11}\gamma_{31} + \gamma_{12}\gamma_{31} + a_{13}\gamma_{33}) = 0,$$
$$R_{15} \equiv a_{11}\gamma_{31} + a_{12}a_{31} + a_{13}\gamma_{33} + \gamma_{11}a_{31} + \gamma_{12}\gamma_{31} + \gamma_{13}a_{33} = 0,$$
$$R_{34} \equiv R_{12},\ R_{36} \equiv R_{26} \equiv R_{45} \equiv R_{15},\quad R_{46} \equiv R_{16},$$
$$R_{16} + R_{25} \equiv (\delta + 1)R_{15},\quad R_{35} \equiv R_{25}.$$

Hence all follow from R_{56}, R_{13}, (s), R_{16}, R_{15}. If in R_{16} and R_{15} we collect the terms in γ_{31}, a_{31}, we find that the determinant of their coefficients is unity by (s). Hence R_{16} and R_{15} serve to determine γ_{31}, a_{31}, which occur in neither R_{56} nor R_{13}. Now R_{56} has $2^n + 1$ sets of solutions a_{33}, γ_{33}; (s) has $2^n + 1$ sets of solutions $\lambda \equiv a_{11} + \gamma_{12}$, $\mu \equiv \gamma_{11} + a_{12}$, not both zero. Then R_{13} becomes

$$\mu a_{11}(\delta + 1) + \lambda\gamma_{11}(\delta + 1) + \lambda\mu + a_{13}^2 + (\delta + 1)a_{13}\gamma_{13} + \gamma_{13}^2 = 0,$$

and hence determines a_{11} or else γ_{11}. But a_{13}, γ_{13} and a_{11} or else γ_{11} remain arbitrary. Hence there are $2^{3n}(2^n + 1)$ transformations.

Then R_{12}, (37), (38), R_{34}, R_{23} become, respectively,

$$\alpha_{11}\delta_{11}-\gamma B=1,\quad \beta_{21}+\beta_{31}=B-\varkappa,\quad \delta_{11}(2\varkappa-B)+B(2\sigma-B)=0, \tag{137}$$

$$2\alpha\alpha_{11}-2\gamma\rho=1+\alpha_{11}\gamma+\alpha_{11}^2,\quad -\alpha B+\delta_{11}\rho=-\alpha_{11}\sigma-\gamma\varkappa. \tag{138}$$

The determinant of the coefficients of α and ρ in the last two equals 2 by $(137)_1$. Hence they determine α, ρ in terms of α_{11}, δ_{11}, γ, B, σ, $\varkappa$.

Let first δ_{11} be given any one of the p^n-1 marks $\neq 0$, while γ, B, σ, β_{11}, β_{12}, β_{33} are given any one of the p^{6n} sets of 6 marks. Then α_{11} is determined by $(137)_1$, and $\varkappa$ by $(137)_3$. Then, as above, α and ρ are determined by (138). Then conditions (136), read in reverse order, determine β_{13}, β_{23}, β_{22}, β_{32}. Then R_{24} determines β_{21}, and $(137)_2$ determines β_{31}. All the conditions have been satisfied.

Let next $\delta_{11}=0$, and give α_{11}, $\varkappa$, β_{13}, β_{31}, β_{33} any values, γ any value $\neq 0$. Then $(137)_1$ gives $B=-\gamma^{-1}$, $(137)_3$ gives $\sigma=-\frac{1}{2}\gamma^{-1}$. Then, as above, (138) determine α and ρ. Next, $(137)_2$ determines β_{21}; while (136), read in reverse order, determine β_{12}, β_{23}, β_{22}, β_{32}. Then R_{24} determines β_{11}.

For $p>2$, the number of Abelian transformations commutative with $S_{0,1,-1}$ is

$$(p^n-1)\,p^{6n}+(p^n-1)\,p^{5n}=p^{5n}(p^{2n}-1).$$

Type $S_{b,1,1}$, $p=2$, $b=0$ or a particular root of $f(\eta)=1$, f defined by (28). Every Abelian transformation commutative with $S_{b,1,1}$ is given by (36), subject to (39), (40) and R_{24}, where now $c=b$.

Let first $b=0$, so that $B=0$ or δ_{11}. For $B=0$,

$$\alpha_{11}\delta_{11}=1,\quad \gamma=\alpha_{11}+\delta_{11},\quad \beta_{12}+\beta_{21}+\beta_{13}+\beta_{31}=0,$$
$$\gamma(\beta_{21}+\beta_{31}+\beta_{23})+\delta_{11}\beta_{22}+\alpha_{11}\beta_{33}=0,$$

together with R_{24}. If $\gamma=0$, then $\alpha_{11}=\delta_{11}=1$, $\beta_{22}=\beta_{33}$, $\beta_{32}=\beta_{23}$; we may take β_{33}, β_{23}, β_{11}, β_{12}, β_{13}, α arbitray, when β_{21} is determined by R_{24}. If $\gamma\neq 0$, we may take β_{33}, β_{22}, β_{21}, β_{12}, β_{13}, α arbitrary, α_{11} any mark $\neq 0$, 1, whence $\alpha_{11}=1/\delta_{11}$, $\beta_{31}=\beta_{12}+\beta_{21}+\beta_{13}$, while β_{23} is determined by the equation above, β_{32} by $B=0$, β_{11} by R_{24}. Hence the case $B=0$ yields $2^{6n}+2^{6n}(2^n-2)$ sets For $B=\delta_{11}$, (39) give

$$\delta_{11}=\alpha_{11}\neq 0,\quad \gamma=\alpha_{11}^{-1}+\alpha_{11},\quad \beta_{12}+\beta_{13}+\beta_{21}+\beta_{31}=\alpha_{11}.$$

We may take β_{11}, β_{12}, β_{13}, β_{23}, β_{32}, β_{33} arbitrary, α_{11} any mark $\neq 0$. Then β_{22} is determined by $B=\delta_{11}$, β_{21} by R_{24}, β_{31} by the third equation above, α by $(40)_1$.

Combining these $(2^n-1)\,2^{6n}$ sets with those for $B=0$, we get* $2\,(2^n-1)\,2^{6n}$ *as the total number for* $b=0$.

Let next $f(b)=1$, so that $b\neq 0$. If $\delta_{11}=0$,

$$B^2=b,\quad \gamma=B^{-1},\quad \alpha_{11}=0 \text{ or } \gamma.$$

We may take $\beta_{12}, \beta_{21}, \beta_{13}, \beta_{23}, \beta_{32}, \beta_{33}$ arbitrary. Then $(39)_1$ determines β_{31}, $B=b^{\frac{1}{2}}$ determines β_{22}, $(40)_1$ determines α, R_{24} determines β_{11}. There result $2\,.\,2^{6n}$ sets. If $\delta_{11}\neq 0$, $(40)_2$ gives

$$(B\delta_{11}^{-1})^2+B\delta_{11}^{-1}=b+b\delta_{11}^{-2}.$$

Hence, by §9, must $f(b+b\delta_{11}^{-2})=0$, whence $f(b\delta_{11}^{-2})=1$. The latter has 2^{n-1} roots δ_{11} in the $GF[2^n]$. To each of these correspond two roots B in the field. Eliminating α_{11} between the last two equations (39) and simplifying by $(40)_2$, we get

$$b\gamma^2+\gamma\delta_{11}=1+\delta_{11}^2.$$

Combining this with $(40)_2$, we get $b\gamma=B$ or $B+\delta_{11}$. Hence there are precisely 2^{n+1} sets α_{11}, $\delta_{11}\neq 0$, B, γ satisfying $(39)_1$, $(39)_2$, $(40)_2$. Of these exactly two sets have $\gamma=0$:

$$\alpha_{11}=\delta_{11}=1,\quad B=0,\quad \gamma=0;\quad \alpha_{11}=\delta_{11}=1,\quad B=1,\quad \gamma=0.$$

For $\gamma=0$, we may take α, β_{23}, β_{33}, β_{11}, β_{12}, β_{13} arbitrary, when β_{22}, β_{32}, β_{21}, β_{31} are determined by $(40)_1$, B, R_{24}, $(39)_1$, respectively. For $\gamma\neq 0$, we may take α, β_{23}, β_{33}, β_{32}, β_{31}, β_{13} arbitrary, when β_{22}, β_{21}, β_{12} β_{11} are determined by B, $(40)_1$, $(39)_1$, R_{24}, respectively. There result $2^{n+1}.\,2^{6n}$ sets for $\delta_{11}\neq 0$. *Hence,*† *there are* $2\,(2^n+1)\,2^{6n}$ *sets for* $f(b)=1$.

Type $S_{0,\mu}T_{1,\pm 1}T_{2,\pm 1}T_{3,\pm 1}$, $p>2$. An Abelian transformation commutative with one of the four forms must have the form (44), subject to (45)–(48), where now $B=\beta'=0$, $A=\alpha=\mu$. Hence

$$\alpha_{11}=\delta_{11}=\pm 1,\quad \alpha_{21}=-\delta_{12},\quad \alpha_{31}=\delta_{12}+\delta_{13},\quad \beta_{31}=\beta_{13}+2\mu\delta_{13}+\mu\delta_{12},$$
$$\beta_{21}=\mu\delta_{13}-\beta_{12}+\beta_{13}^{\,\prime},\quad 2\delta_{13}=\pm\,\delta_{12}^2-\delta_{12},$$

together with (48), which determines β_{12}. Since β_{11}, δ_{12}, β_{13} remain arbitrary, there are $2p^{3n}$ transformations.

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* I note for check that I had obtained this number as the number of Abelian transformations commutative with $V_{0,1,0}$ which (§15) is conjugate with $S_{0,1,1}$.

† Obtained also by a different but longer method of solution.

Type $W_{0,\beta}$, $p=2$, $\beta=0$ or a particular root of $f(\eta)=1$. An Abelian transformation commutative with $W_{0,\beta}$ must have the form (50), subject to (51) and (52), where $D=\delta=0$, $B=\beta$. The $2 \, . \, 2^{3n}$ sets are given as follows

$$\alpha_{11}=0, \quad \alpha_{12}=1, \quad \delta_{13}=\beta_{13}+\beta_{13}^{\dagger}, \quad \beta_{11}, \beta_{12}, \beta_{13} \text{ arbitrary};$$
$$\alpha_{11}=1, \quad \alpha_{12}=0, \quad \beta_{13}=\delta_{13}+\delta_{13}^{\dagger}, \quad \beta_{11}, \beta_{12}, \delta_{13} \text{ arbitrary}.$$

Type $T_{1,\kappa}\,T_{2,\kappa^{p^n}}\,T_{3,\kappa^{p^{2n}}}$, $\kappa^{p^{3n}-1}=1$, $\kappa^{p^n-1}\neq 1$, furnished by (90). The six multipliers are all distinct* by §25. The replacement of κ by κ^{p^n} is effected by transforming by $(\xi_1\xi_2\xi_3)(\eta_1\eta_2\eta_3)$; the replacement of κ by κ^{-1}, by $M_1M_2M_3$. Hence there are $\frac{1}{6}\,(p^{3n}-p^n)$ non-conjugate forms. Each is commutative with $p^{3n}-1$ Abelian transformations having simultaneously the canonical form $T_{1,a}\,T_{2,a^{p^n}}$, $T_{3,a^{p^{2n}}}$, $a^{p^{3n}-1}=1$.

Type $T_{1,\kappa}\,T_{2,\kappa^{p^n}}\,T_{3,\kappa^{p^{2n}}}$, $\kappa^{p^{6n}}=\kappa$, $\kappa^{p^{3n}}\neq\kappa$, $\kappa^{p^{2n}}\neq\kappa$, furnished by (91). Now $\kappa^{-1}=\kappa^{p^{3n}}$ by §26, end. Of the $p^{3n}+1$ solutions, it is only necessary to exclude those making $\kappa^{p^n+1}=1$. Hence, as for preceding type, there are $\frac{1}{6}\,(p^{3n}-p^n)$ non-conjugate forms. Each is commutative with $p^{3n}+1$.

Type (100) for $\sigma=1/(\kappa^{-1}-\kappa)$, $\kappa^{p^n+1}=1$, $\kappa^2\neq 1$. Now $M_1P_{23}M_2M_3=P$ transforms (100) into a transformation which may be derived directly from (100) by replacing κ by κ^{-1}, σ by $\bar{\sigma}$, throughout. Now the transformer P does not preserve the conjugacy of variables since, for it, $\bar{x}_i=-y_i$. Now the transformation $x_i'=x_i$, $y_i'=-y_i\,(i=1, 2, 3)$ multiplies the bilinear function $\Sigma\,(x_iy_i^*-y_ix_i^*)$ by -1. But by the interchange of κ and κ^{-1}, the sign of the second member (101) is altered. Hence if we return from the canonical variables x_i, y_i to the original variables ξ_i, η_i, $(100)_\kappa$ and $(100)_{\kappa^{-1}}$ become transformations conjugate within the special Abelian group. Hence there are exactly $\frac{1}{2}\,(p^n-1)$ or $\frac{1}{2}\,2^n$ non-conjugate forms of the present type. If we determine the general ternary tarnsformation S commutative with the transformation which (100) effects on x_1, x_2, x_3, and take the conjugate transformation $\bar{S}$ on y_1, y_3, y_2, we obtain as the

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* To prove directly, if $\kappa^{-1}=\kappa^{p^n}$, then $\kappa^{p^{2n}}=\kappa^{-p^n}=\kappa$, whence $\kappa=\kappa^{p^n}$. If $\kappa^{-1}=\kappa^{p^{2n}}$, then $\kappa=\kappa^{p^{3n}}=\kappa^{-p^n}$, contrary to the preceding.

most general transformation commutative with (100)

$$\begin{pmatrix} a & 0 & b & 0 & 0 & 0 \\ 0 & a^{p^n} & 0 & 0 & 0 & b^{p^n} \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & -b^{p^n} & 0 & a^{p^n} & 0 & c^{p^n} \\ -b & 0 & c & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a^{p^n} \end{pmatrix}.$$

The Abelian conditions on it reduce to the three:

$$aa^{p^n} = 1, \quad -ab^{p^n} + ba^{p^n} = 0, \quad -bb^{p^n} - a^{p^n}c - ac^{p^n} = 0.$$

Hence b/a belongs to the $GF[p^{2n}]$. The third condition may be written

$$\left(\frac{c}{a}\right)^{p^n} + \frac{c}{a} = b^{p^n+1}.$$

Subtracting it from the power of p^n it, we see that it has p^n solutions c/a in the $GF[p^{2n}]$, when b belongs to that field. Hence* there are exactly $p^{2n}(p^n+1)$ transformations in G which are commutative with one having the canonical form (100).

Checks. Table for $p^n = 2$, $m = 3$.

42. The complete list of canonical types for $m = 3$ is given in §§37-41, together with the number N of non-conjugate canonical forms of each type and the number C of transformations of the group G which are commutative with each form. The order of G for the $GF[p^n]$ is

$$\Omega \equiv p^{9n}(p^{6n} - 1)(p^{4n} - 1)(p^{2n} - 1).$$

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* As a check, I verified that the only Abelian substitutions in the GF [2] commutative with (75) for $a = \delta = 1$, and necessarily $\gamma = g = 1$, are the 12:

$$\begin{pmatrix} a_{11} & \gamma_{11} & a_{12} & 0 & a_{13} & 0 \\ \gamma_{11} & a_{11} + \gamma_{11} & a_{12} + a_{13} + \gamma_{11} & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} + \gamma_{11} & 0 & \gamma_{11} & 0 \\ a_{13} & a_{12} & \beta_{22} & a_{11} & \beta_{23} & \gamma_{11} \\ 0 & 0 & \gamma_{11} & 0 & a_{11} & 0 \\ a_{12} & a_{12} + a_{13} + \gamma_{11} & a_{12} + \beta_{22} + \beta_{23} & \gamma_{11} & \beta_{22} & a_{11} + \gamma_{11} \end{pmatrix},$$

where, for $\gamma_{11} = 0$, we get $a_{11} = 1$, $a_{12} = 0$, $\beta_{22} = a_{13}$, β_{23} being arbitrary; for $\gamma_{11} = 1$,

$$a_{13} = a_{11} + a_{11}a_{12}, \quad \beta_{23} = a_{11}a_{12} + a_{11}\beta_{22} + a_{12}.$$

Further, I verified that (75) is now of period 12.

One check on the work is that C is found to divide Ω for each type. A more fundamental check is furnished by equality of Ω and the sum of the expressions $N\Omega \div C$ (the total number of transformations reducible to a given *type*) for the complete list of types. This lengthy computation was made for $p > 2$ with the greatest care, each step being checked.* The resulting polynomial of degree 21 in p^n was found to be identical with Ω, thus furnishing a 21-fold check. This check was particularly convincing as the individual expressions to be added involved fractions which, in their "lowest terms," had as denominators the various factors of 48.

The case $p = 2$ was checked only when $n = 1$ (see accompanying table, the total in the last column being $1451520 = 2^9.3^4.5.7 = \Omega$). In fact, the separation of the cases $p = 2$, $p > 2$ is of trivial character except for a few types.

Independent of the preceding checks, I convinced myself that no two distinct canonical types in the list represented transformations conjugate within G.

* The expressions were left in factored forms so that many sets could be readily added. Those in §§39–41 were twice added with the same result.

Table of non-conjugate types of operators of $SA(6, 2)$.

Type	Period	Commutative	Conjugate
Identity	1	all	1
$L_{11}L_{21}L_{31}$	2	$2^7.3$	3780
L_{11}	2	$2^9.3^2.5$	63
$L_{11}L_{21}$	2	$2^9.3$	945
R_{121}	2	$2^9.3^2$	315
$R_0L'_{31}$	4	2^5	45360
R_0	4	$2^6.3$	7560
R_1	4	$2^6.3$	7560
$T_{2,\lambda}(\lambda^2=\lambda+1)$	3	$2^4.3^3.5$	672
$L_{11}T_{2\lambda}$	6	$2^4.3^2$	10080
$L_{11}T_{2\lambda}L_{31}$	6	$2^4.3$	30240
$T_{1\sigma}T_{2\sigma^2}(\sigma^5=1)$	5	2.3.5	48384
$T_{1\sigma}T_{2\sigma^2}L_{31}$	10	2.5	145152
$T_{1\kappa}T_{2\kappa}(\kappa^2=\kappa+1)$	3	$2^2.3^3$	13440
$T_{1\kappa}T_{2\kappa}L_{31}$	6	$2^2.3^2$	40320
$(63)\ L_{11}$	6	$2^2.3$	120960
(63)	6	$2^2.3^2$	40320
$T_{1\sigma}T_{2\sigma^2}T_{3\lambda}$	15	3.5	96768
$T_{1\lambda}T_{2\lambda}T_{3\lambda}$	3	$2^3.3^4$	2240
$(63)\ T_{3\kappa}$	6	$2^3.3^2$	20160
$R_0T_{3\lambda}$	12	$2^3.3$	60480
$R_1T_{3\lambda}$	12	$2^3.3$	60480
$R_{121}T_{3\lambda}$	6	$2^4.3^2$	10080
S_{111}	4	$2^7.3$	3780
S_{011}	4	2^7	11340
W_{00}	8	2^4	90720
W_{01}	8	2^4	90720
$T_{1\rho}T_{2\rho^2}T_{3\rho^4}(\rho^7=1)$	7	7	207360
$T_{1\rho}T_{2\rho^2}T_{3\rho^4}(\rho^9=1)$	9	3^2	161280
$(100),\ (\kappa^3=1,\ \sigma=1)$	12	$2^2.3$	120960

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Invariants of a System of Linear Partial Differential Equations, and the Theory of Congruences of Rays.

By E. J. Wilczynski.*

In a series of papers, published in the Transactions of the American Mathematical Society, the author has laid the foundations for a theory of ruled surfaces, based upon the consideration of the invariants of a system of ordinary linear differential equations. In the present paper, these considerations are extended to partial differential equations. The paper confines itself, for the most part, to a special case, and this for two reasons. In the first place, the complication of the more general cases is such as to make it appear unwise to attempt their treatment at present; moreover, the real essentials of the method appear just as well in a special case. In the second place, the special problem considered has its own intrinsic interest, as it leads to a theory of congruences. A whole geometry seems to be possible on this basis. The theory of curves, of surfaces, of complexes, etc., can all be built up, and have in part already been built up, by considerations which are of the same character as those discussed in this paper.

§1. *Determination of the most general transformation which converts a general system of linear partial differential equations of the first order into another of the same kind.*

Let us consider a system of q homogeneous linear partial differential equations of the first order

$$\sum_{k=1}^{n}\sum_{i=1}^{m} a_{ki}^{(\mu)} \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{n} b_k^{(\mu)} y_k = 0, \quad (\mu = 1, 2, \ldots . q) \tag{1}$$

with the n unknown functions $y_1, \ldots . y_n$ of the m independent variables

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$x_1, \ldots . x_m$. We wish to find the most general point transformation

$$y_k = g_k(\xi_1, \ldots . \xi_m; \eta_1, \ldots . \eta_n), \quad x_l = f_l(\xi_1, \ldots . \xi_m; \eta_1, \ldots . \eta_n),$$
$$(k = 1, 2, \ldots . n), \quad (l = 1, 2, \ldots . m), \tag{2}$$

which will transform (1) into another system of the same form.

We have from (2)

$$dy_k = \sum_{\lambda=1}^{m} \frac{\partial g_k}{\partial \xi_\lambda} d\xi_\lambda + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} d\eta_\mu,$$
$$dx_k = \sum_{\lambda=1}^{m} \frac{\partial f_k}{\partial \xi_\lambda} d\xi_\lambda + \sum_{\mu=1}^{n} \frac{\partial f_k}{\partial \eta_\mu} d\eta_\mu.$$

But after the transformation η_k is a function of $\xi_1, \ldots . \xi_m$, so that

$$d\eta_k = \sum_{\nu=1}^{m} \frac{\partial \eta_k}{\partial \xi_\nu} d\xi_\nu,$$

and therefore

$$dy_k = \sum_{\lambda=1}^{m} \left(\frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) d\xi_\lambda,$$
$$dx_k = \sum_{\lambda=1}^{m} \left(\frac{\partial f_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial f_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) d\xi_\lambda.$$

But we also have

$$dy_k = \sum_{\nu=1}^{m} \frac{\partial y_k}{\partial x_\nu} dx_\nu,$$

whence, substituting the above values for dy_k and dx_ν,

$$dy_k = \sum_{\lambda=1}^{m} \left(\frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) d\xi_\lambda = \sum_{\nu=1}^{m} \left[\frac{\partial y_k}{\partial x_\nu} \sum_{\lambda=1}^{m} \left(\frac{\partial f_\nu}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial f_\nu}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) \right] d\xi_\lambda,$$
$$(k = 1, 2, \ldots . n).$$

These equations must be satisfied for all values of $d\xi_1, \ldots . d\xi_m$, so that we must have

$$\frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} = \sum_{\nu=1}^{m} \frac{\partial y_k}{\partial x_\nu} \left(\frac{\partial f_\nu}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial f_\nu}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right), \tag{3}$$
$$(k = 1, 2, \ldots . n; \lambda = 1, 2, \ldots . m),$$

a system of mn equations whose solution would give the expressions for $\frac{\partial y_k}{\partial x_\nu}$ in terms of the quantities $\frac{\partial \eta_k}{\partial \xi_\nu}$. These expressions must be linear if the transformed

of system (1) is to be again a linear system. But this will be so if, and only if, all of the partial derivatives $\frac{\partial f_\nu}{\partial \eta_\mu}$ vanish, i. e. if the f_ν quantities are functions of $\xi_1, \dots \xi_m$ alone. Equations (3) then become

$$\sum_{\nu=1}^{m} \frac{\partial f_\nu}{\partial \xi_\lambda} \frac{\partial y_k}{\partial x_\nu} = \frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda}, \tag{3a}$$

where now obviously the right members must be linear functions of $\eta_1, \dots \eta_n, \frac{\partial \eta_k}{\partial \xi_1}$, etc. But this is so only if

$$\frac{\partial g_k}{\partial \eta_\mu} = \alpha_{k\mu}$$

is a function of $\xi_1, \dots \xi_m$ alone, i. e. if

$$g_k = \alpha_{k0} + \alpha_{k1} y_1 + \dots + \alpha_{kn} y_n.$$

But the right member of (3a) must be linear and *homogeneous* in $\eta_1, \dots \eta_n$, $\frac{\partial \eta_1}{\partial \xi_1}$, etc. Therefore

$$\frac{\partial \alpha_{k0}}{\partial \xi_\lambda} = 0, \qquad (\lambda = 1, 2, \dots\dots m;\ k = 1, 2, \dots n),$$

i. e. α_{k0} can only be a constant. But it must even be zero, for else the new system obtained from (1) by the transformation, while again linear, would not be homogeneous.

We have, therefore, the following result: *The most general transformation which converts system* (1) *into another of the same form, is homogeneous and linear in the dependent variables, but entirely arbitrary in the independent variables. The general form of the transformation is*

$$y_k = \alpha_{k1}\eta_1 + \alpha_{k2}\eta_2 + \dots + \alpha_{kn}\eta_n, \qquad (k = 1, 2, \dots n),$$
$$x_l = f_l(\xi_1, \xi_2, \dots \xi_m), \qquad (l = 1, 2, \dots m),$$

where α_{ki} and f_l are arbitrary functions of $\xi_1, \dots \xi_m$, subject of course to the restriction that the determinant

$$|\alpha_{ki}|$$

and the Jacobian of $f_1, \dots f_m$ with respect to $\xi_1, \dots \xi_m$ shall not vanish.

This theorem applies of course only to a *general* system of form (1) and to

transformations which have no special relation to the system considered. The same remark applies to Staeckel's proof of the corresponding theorem for a single ordinary linear differential equation* as well as to my own proof of the corresponding theorem for a system of such equations.† It is easy enough to see that the theorem may be extended to cover systems of partial differential equations containing higher derivatives than the first, moreover not necessarily linear but still homogeneous. All of these results are simple consequences of Lie's theory.

§2.—*Introduction of the special problem. General remarks on the calculation of the invariants.*

We shall, in this paper, confine our attention exclusively to the special case of two equations with two dependent and two independent variables, viz.

$$\left.\begin{aligned}\Omega &= ay_1 + by_2 + cz_1 + dz_2 + ey + fz = 0,\\ \Omega' &= a'y_1 + b'y_2 + c'z_1 + d'z_2 + e'y + f'z = 0,\end{aligned}\right\} \tag{1}$$

where

$$y_k = \frac{\partial y}{\partial x_k}, \quad z_k = \frac{\partial z}{\partial x_k}, \quad (k = 1, 2) \tag{2}$$

and where $a, b, \ldots . f'$ are functions of x_1 and x_2.

Let $\alpha, \beta, \gamma, \delta, g$ and h be arbitrary functions of ξ_1 and ξ_2 such that the determinants $\alpha\delta - \beta\gamma$ and $\dfrac{\partial(g, h)}{\partial(\xi_1, \xi_2)}$ do not vanish. Then, as we have seen in §1, the most general transformation which converts (1) into a system of the same character is of the form

$$\left.\begin{aligned} y &= \alpha\eta + \beta\zeta, & z &= \gamma\eta + \delta\zeta;\\ x_1 &= g(\xi_1, \xi_2), & x_2 &= h(\xi_1, \xi_2).\end{aligned}\right\} \tag{3}$$

But clearly one may also consider instead of (1), any system of the form

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0 \tag{4}$$

for these two systems are entirely equivalent. Since the coefficients of (4) are simple combinations of those of (1), the operation of replacing (1) by (4) may also be looked upon as a transformation.

In calculating the invariants of (1) we shall apply these various transforma-

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* Staeckel, Crelle's Journal, Vol. 111.

† Wilczynski, Am. Jour. of Math., 1901.

tions successively. We shall call those functions of the coefficients of (1) which remain invariant under the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \tag{5}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x_1 and x_2, *pseudo-invariants.* Those functions of pseudo invariants, which are invariant under the additional transformation which corresponds to the substitution of (4) in place of (1), shall be called *seminvariants.* Finally, those functions of seminvariants, which do not change when the independent variables are arbitrarily transformed, shall be called *invariants.* A similar nomenclature is to be applied to the covariants.

§3.—*Determination of the pseudo-invariants.*

If we apply transformation (5) to system (1), a new system

$$\bar{\Omega} = 0, \quad \bar{\Omega}' = 0$$

is obtained, whose coefficients are

$$\left.\begin{aligned} \bar{a} &= a\alpha + c\gamma, \quad \bar{c} = a\beta + c\delta, \\ \bar{b} &= b\alpha + d\gamma, \quad \bar{d} = b\beta + d\delta, \\ \bar{e} &= a\alpha_1 + b\alpha_2 + c\gamma_1 + d\gamma_2 + e\alpha + f\gamma, \\ \bar{f} &= a\beta_1 + b\beta_2 + c\delta_1 + d\delta_2 + e\beta + f\delta, \end{aligned}\right\} \tag{6}$$

while $\bar{a}'$, $\bar{b}'$, etc., are given by the same equations if every letter a, b, c, etc., be replaced by a', b', c', etc.

The transformations (5) form an infinite group, which may be defined by differential equations (in the sense of Lie). Therefore, Lie's theory of such groups may be applied. The most general infinitesimal transformation of this group will be obtained by putting

$$\alpha = 1 + \lambda\delta t, \quad \beta = \mu\delta t, \quad \gamma = \nu\delta t, \quad \delta = 1 + \rho\delta t,$$

where δt is an infinitesimal and where λ, μ, ν, ρ are arbitrary functions of x_1 and x_2. The corresponding infinitesimal variations of a, b, c, etc., may be easily computed from (6). They are

$$\left.\begin{aligned} \delta a &= (\lambda a + \nu c)\,\delta t, \quad \delta c = (\mu a + \rho c)\,\delta t, \\ \delta b &= (\lambda b + \nu d)\,\delta t, \quad \delta d = (\mu b + \rho d)\,\delta t, \\ \delta e &= (\lambda_1 a + \lambda_2 b + \nu_1 c + \nu_2 d + \lambda e + \nu f)\,\delta t, \\ \delta f &= (\mu_1 a + \mu_2 b + \rho_1 c + \rho_2 d + \mu e + \rho f)\,\delta t, \\ &\text{etc.} \qquad\qquad \text{etc.} \end{aligned}\right\} \tag{7}$$

If any function F of these quantities is an absolute invariant under transformation (5), δF must vanish for arbitrary values of λ, μ, ν, ρ and of their various derivatives. The system of partial linear differential equations thus obtained for F is always a complete system according to Lie's theory. Although we shall be able to obtain most of our invariants without integrating such systems, we shall always use them to furnish the proof of the completeness of the system of invariants obtained otherwise.

From equations (6) it is obvious that

$$\left.\begin{aligned} i &= ad - bc\,, & i' &= a'd' - b'c',\\ k &= ac' - a'c, & l &= bd' - b'd,\\ m &= ad' - b'c, & n &= a'd - bc'\,, \end{aligned}\right\} \tag{8}$$

are relative pseudo-invariants, their quotients being absolute pseudo-invariants. Each of these functions satisfies the equation

$$\bar{F} = (\alpha\delta - \beta\gamma)\, F.$$

These pseudo-invariants involve merely the eight coefficients a, b, c, d; a', b', c', d'. If we set up the complete system of partial differential equations satisfied by pseudo-invariants of this kind, we find that it contains 4 independent equations and 8 variables. Therefore, there are only four independent absolute, and only five relative pseudo-invariants depending on these variables.

In fact, there is a syzygy between the quantities (8), namely,

$$ii' - mn - kl = 0, \tag{9}$$

and we shall take

$$i,\ i',\ k,\ l \text{ and } t = m - n \tag{10}$$

as our system of pseudo-invariants, for these five are independent. That this is so may be easily seen if we put

$$b' = 0, \quad c' = d' = 1.$$

For then

$$i = ad - bc, \quad i' = a', \quad k = a - a'c, \quad l = b, \quad t = a + b - a'd,$$

and the Jacobian of these five functions with respect to the five variables a, b, c, d, a', disregarding sign, is

$$a'(a - b + a'd),$$

and this does not vanish identically.

If we set up the system of partial differential equations, satisfied by the pseudo-invariants depending on the quantities

$$\left.\begin{array}{l} a\,, \ldots . d\,;\; a'\,, \ldots . d'\,;\; e\,, f\,; \\ a_k, \ldots . d_k;\; a'_k, \ldots . d'_k;\; e', f';\; (k=1,\,2), \end{array}\right\} \tag{11}$$

we find that it consists of twelve independent equations with 28 independent variables. There are therefore 16 independent absolute or 17 relative pseudo-invariants containing these coefficients and their first derivatives.

In order to construct them we put

$$\left.\begin{array}{ll} p = e - a_1 - b_2\,, & p' = e' - a'_1 - b'_2\,, \\ q = f - c_1 - d_2\,, & q' = f' - c'_1 - d'_2\,. \end{array}\right\} \tag{12}$$

It is easy to see that the pairs of quantities

$$p,\, q;\quad p',\, q';\quad a,\, c;\quad b,\, d;\quad a',\, c';\quad b',\, d';$$

are all transformed by the same equations, i. e. as we shall say are *cogredient*. Therefore the determinant of any two of these pairs is a pseudo-invariant.

Let us put

$$\left.\begin{array}{ll} r = aq - cp\,, & r' = a'q' - c'p'\,, \\ s = bq - dp\,, & s' = b'q' - d'p'\,. \end{array}\right\} \tag{13}$$

Then these quantities are pseudo-invariants.

Moreover if ϕ and ψ are two pseudo-invariants, for which

$$\bar{\phi} = \Delta^n \phi, \quad \bar{\psi} = \Delta^n \psi, \quad \Delta = \alpha\delta - \beta\gamma.$$

we have

$$\frac{\bar{\phi}_k}{\bar{\phi}} - \frac{\bar{\psi}_k}{\bar{\psi}} = \frac{\phi_k}{\phi} - \frac{\psi_k}{\psi}, \qquad (k=1,\,2),$$

where

$$\phi_k = \frac{\partial \phi}{\partial x_k}, \quad \psi_k = \frac{\partial \psi}{\partial x_k}.$$

Therefore,

$$\overline{\phi_k \psi - \phi \psi_k} = \Delta^{2n} (\phi_k \psi - \phi \psi_k). \tag{14}$$

Let us denote $\phi_k\psi - \phi\psi_k$ by $(\phi_k,\, \psi)$ and call it the *Wronskian* of ϕ and ψ with respect to x_k. It is clear then that the Wronskians of two such pseudo-invariants are again pseudo-invariants.

We can, therefore, write down the following list of 17 pseudo-invariants, all of which contain only the 28 quantities (11):

$$i,\, i',\, k,\, l,\, t;\; (i_k,\, t),\, (i'_k,\, t),\, (k_k,\, t),\, (l_k,\, t);\; r,\, s,\, r',\, s'; \qquad (k=1,\,2). \tag{15}$$

They are independent. For the first five are independent of each other, as we have already seen. Each of the next eight contains a quantity which is not contained in any preceding one, so that the first thirteen are independent. The last four contain the additional variables e, f, e', f' in four independent combinations, so that there can be no relation between the seventeen pseudo-invariants (15).

We continue our search for pseudo-invariants by asking for all those which contain, besides the variables (11), the second derivatives of $a \ldots . d$, $a' \ldots . d'$, and the first derivatives of e, f, e', f', i. e. altogether sixty quantities. The system of partial differential equations will consist, in this case, of twenty-four equations, obtained by equating to zero in δF the coefficients of λ, μ, ν, ρ and of the first and second derivatives of these functions with respect to x_1 and x_2. These twenty-four equations are independent. In order to abbreviate the proof of this statement, let us represent the matrix of the fourth order:

$$\begin{matrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{matrix}$$

by M, and a matrix of the fourth order all of whose terms are zero, by 0. If we now write down the matrix of the coefficients of our system of partial differential equations, we find in it the following determinant of the 24th order

$$\begin{vmatrix} 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & M \\ 0 & M & 0 & 2M_1 & M_2 & 0 \\ 0 & 0 & M & 0 & M_1 & 2M_2 \\ M & M_1 & M_2 & M_{11} & M_{12} & M_{22} \end{vmatrix}$$

where M_1 is the same matrix as M formed with a_1, b_1, etc. This determinant does not vanish. Therefore the 24 equations are independent.

This system of 24 equations must therefore have $60 - 24 = 36$ independent solutions, i. e. there must be 36 absolute, or 37 relative pseudo-invariants depending on these 60 variables.

We can easily form the twenty new pseudo-invariants. According to (14), the sixteen Wronskians

$$\left.\begin{array}{l}((i_1, t)_1, t^2), \quad ((i_1', t)_1, t^2), \quad ((k_1, t)_1, t^2), \quad ((l_1, t)_1, t^2), \\ ((i_2, t)_2, t^2), \text{ etc.,} \\ ((i_1, t)_2, t^2), \text{ etc.,} \\ ((i_2, t)_1, t^2), \text{ etc.,}\end{array}\right\} \tag{16}$$

are such pseudo-invariants. But only twelve of these are independent, for we have

$$((i_2, t)_1, t^2) = ((i_1, t)_2, t^2), \text{ etc.,} \tag{17}$$

as may be seen by direct computation.

The following eight quantities

$$\left.\begin{array}{llll}(r_1, t), & (r_2, t), & (r_1', t), & (r_2', t), \\ (s_1, t), & (s_2, t), & (s_1', t), & (s_2', t),\end{array}\right\} \tag{18}$$

are also pseudo-invariants. Together, we now have the twenty new pseudo-invariants. For, they are independent of the previous seventeen, and of each other, as each of them contains a variable not contained in any previous one, viz.

$$\begin{array}{llll} i_{11}, & i_{11}', & k_{11}, & l_{11}, \\ i_{22}, & i_{22}', & k_{22}, & l_{22}, \\ i_{12}, & i_{12}', & k_{12}, & l_{12}, \\ af_1 - ce_1, & af_2 - ce_2, & a'f_1' - c'e_1', & a'f_2' - c'e_2', \\ bf_1 - de_1, & bf_2 - de_2, & b'f_1' - d'e_1', & b'f_2' - d'e_2', \end{array}$$

where the last eight are independent combinations of the eight variables $e_1, \ldots . f_2'$.

Of course this process for obtaining pseudo-invariants can be continued. There are 65 relative pseudo-invariants containing no higher derivatives of a, $b, \ldots . a'$ than the third, and no higher derivatives of $e, \ldots . f'$ than the second. For, the corresponding system of partial differential equations consists of 40 equations with 104 variables. These pseudo-invariants can all be obtained by continuing the Wronskian processes in an obvious manner. A simple induction completes the proof of the following theorem.

The pseudo-invariants of (1) *are all functions of* $i, i', k, l, t, r, s, r', s'$ *and of the quantities obtained by combining these with powers of* t *according to the Wronskian process.*

§4.—*Determination of the seminvariants.*

Let us now consider the system

$$\left.\begin{array}{l}\phi\Omega + \psi\Omega' = 0, \\ \chi\Omega + \omega\Omega' = 0,\end{array}\right\} \tag{19}$$

instead of (1), to which it is obviously equivalent if

$$\Delta = \phi\omega - \psi\chi \neq 0, \tag{20}$$

the functions ϕ, ψ, χ, ω of x_1 and x_2 being otherwise arbitrary. Then system (19) is of the same form as (1), and denoting its coefficients by capital letters, we have

$$\left.\begin{aligned} A &= \phi a + \psi a', & A' &= \chi a + \omega a', \\ B &= \phi b + \psi b', & B' &= \chi b + \omega b', \\ C &= \phi c + \psi c', & C' &= \chi c + \omega c', \\ D &= \phi d + \psi d', & D' &= \chi d + \omega d', \\ E &= \phi e + \psi e', & E' &= \chi e + \omega e', \\ F &= \phi f + \psi f', & F' &= \chi f + \omega f'. \end{aligned}\right\} \tag{21}$$

We can therefore regard the process of replacing system (1) by (19) as a transformation upon the coefficients of (1) by means of the equations (21), which are obviously again the equations of an infinite continuous group which can be defined by differential equations.

Seminvariants are such functions of the coefficients of system (1) and of their derivatives which are invariant under this transformation besides being pseudo-invariants.

Denoting systematically the various quantities formed for the new system by capital letters, we easily find

$$\left.\begin{aligned} I &= \phi^2 i + \phi\psi(m+n) + \psi^2 i', \\ I' &= \chi^2 i + \chi\omega(m+n) + \omega^2 i', \\ K &= \Delta k, \quad L = \Delta l, \quad T = \Delta t, \end{aligned}\right\} \tag{22}$$

so that k, l and t are at once seen to be seminvariants, as are also the combinations (k_i, t) and (l_i, t), and all others obtained by a repetition of this process.

We again introduce infinitesimal transformations by putting

$$\phi = 1 + \varepsilon\delta t, \quad \psi = \zeta\delta t, \quad \chi = \eta\delta t, \quad \omega = 1 + \theta\delta t,$$

where ε, ζ, η, θ are arbitrary functions and δt is an infinitesimal. From (22) we then find immediately

$$\left.\begin{aligned} \delta i &= 2\varepsilon i + \zeta(m+n), \\ \delta i' &= \eta(m+n) + 2\theta i', \\ \delta k &= (\varepsilon+\theta)k, \\ \delta l &= (\varepsilon+\theta)l, \\ \delta t &= (\varepsilon+\theta)t, \end{aligned}\right\} \tag{23}$$

where the infinitesimal factor δt has been omitted everywhere on the right members.

Remembering that

$$(i_k, t) = i_k t - i t_k, \quad ((m+n)_k, t) = (m_k + n_k)\, t - (m+n)\, t_k, \text{ etc.,}$$

we find from (23), again omitting the infinitesimal factor,

$$\left.\begin{aligned}
\delta(i_k, t) &= 3\varepsilon(i_k, t) + \zeta((m+n)_k, t) + \theta(i_k, t) + \zeta_k t(m+n) - \theta_k i t, \\
\delta(i'_k, t) &= \varepsilon(i'_k, t) + \eta((m+n)_k, t) + 3\theta(i'_k, t) + \eta_k t(m+n) - \varepsilon_k i' t, \\
\delta(k_k, t) &= 2(\varepsilon+\theta)(k_k, t), \qquad (k = 1, 2). \\
\delta(l_k, t) &= 2(\varepsilon+\theta)(l_k, t),
\end{aligned}\right\} \tag{24}$$

From equations (12) and (13) we find

$$\left.\begin{aligned}
\delta r &= 2\varepsilon r + \zeta u - \zeta_1 k - \varepsilon_2 i - \zeta_2 m, \\
\delta s &= 2\varepsilon s + \zeta v + \varepsilon_1 i + \zeta_1 n - \zeta_2 l, \\
\delta r' &= 2\theta r' + \eta u + \eta_1 k - \eta_2 n - \theta_2 i', \\
\delta s' &= 2\theta s' + \eta v + \eta_1 m + \theta_1 i' + \eta_2 l,
\end{aligned}\right\} \tag{25}$$

where we have put

$$\left.\begin{aligned}
u &= aq' - cp' + a'q - c'p, \\
v &= bq' - dp' + b'q - d'p,
\end{aligned}\right\} \tag{26}$$

which quantities are manifestly pseudo-invariants which could be expressed algebraically in terms of the standard set.

Let us make a table of these infinitesimal transformations. At the head of each of the seventeen columns we place one of the functions

$$i, i', k, l, t; \quad (i_k, t), (i'_k, t), (k_k, t), (l_k, t); \quad r, s, r', s'.$$

To the left of each horizontal row we write one of the letters ε, ζ, η, θ, ε_1, etc. The infinitesimal transformation of any of the seventeen quantities i, i', etc., is then easily obtained by looking at the corresponding column in the table. Each of the quantities in that column is multiplied by the arbitrary function to the left of the row in which it is found, and the sum of all such products is the infinitesimal transformation required.

But this table, at the same time, serves another purpose. It gives the coefficients of the system of partial differential equations, whose solutions are the seminvariants depending upon the seventeen variables considered. For example, the quantities in the first row of the table are the coefficients of the equation

$$2i\frac{\partial F}{\partial i} + k\frac{\partial F}{\partial k} + l\frac{\partial F}{\partial l} + t\frac{\partial F}{\partial t} + 3(i_1, t)\frac{\partial F}{\partial(i_1, t)} + \text{etc.} = 0,$$

which is obtained by equating to zero the coefficients of ε in the general expression for δF. We may then speak of equation ε, equation ζ, etc.

The table now follows:

	i	i'	k	l	t	(i_1, t)	(i_2, t)	(i_1', t)	(i_2', t)	(k_1, t)	(k_2, t)	(l_1, t)	(l_2, t)	r	s	r'	s'
ε	$2i$	0	k	l	t	$3(i_1, t)$	$3(i_2, t)$	(i_1', t)	(i_2', t)	$2(k_1, t)$	$2(k_2, t)$	$2(l_1, t)$	$2(l_2, t)$	$2r$	$2s$	0	0
ζ	$m+n$	0	0	0	0	(m_1+n_1, t)	(m_2+n_2, t)	0	0	0	0	0	0	u	v	0	0
η	0	$m+n$	0	0	0	0	0	(m_1+n_1, t)	(m_2+n_2, t)	0	0	0	0	0	0	u	v
θ	0	$2i'$	k	l	t	(i_1, t)	(i_2, t)	$3(i_1', t)$	$3(i_2', t)$	$2(k_1, t)$	$2(k_2, t)$	$2(l_1, t)$	$2(l_2, t)$	0	0	$2r'$	$2s'$
ε_1	0	0	0	0	0	0	0	$-i't$	0	0	0	0	0	0	i	0	0
ζ_1	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	0	$-k$	n	0	0
η_1	0	0	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	0	k	m
θ_1	0	0	0	0	0	$-it$	0	0	0	0	0	0	0	0	0	0	i'
ε_2	0	0	0	0	0	0	0	0	$-i't$	0	0	0	0	$-i$	0	0	0
ζ_2	0	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	$-m$	$-l$	0	0
η_2	0	0	0	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	$-n$	l
θ_2	0	0	0	0	0	0	$-it$	0	0	0	0	0	0	0	0	$-i'$	0

Let us consider the ten equations which are obtained by omitting equations ε and θ. None of these ten equations contains the partial derivatives of F with respect to any of the seven quantities k, l, t, (k_i, t), (l_i, t). These ten equations contain only the derivatives with respect to the other ten variables. But the determinant of these equations may be evaluated without much trouble. Disregarding its sign, it is equal to

$$t^4(m+n)^2 i^2 i'^2 (t^2 - 4kl + 3ii')^2,$$

and therefore, in general, different from zero. But this shows that the partial derivatives of F with respect to the ten variables here considered must be zero. In other words, these seminvariants must be functions of k, l, t, (k_i, t), (l_i, t) alone. These latter quantities are themselves seminvariants. Therefore:

All seminvariants which contain no higher derivatives of the coefficients $a, b, \ldots . d'$ than the first and which contain no derivatives of the coefficients e, f, e', f' are functions of the following seven:

$$k, l, t; \quad (k_i, t), (l_i, t); \qquad (i = 1, 2).$$

If we were to continue this method of determining the other seminvariants one step farther, our system of partial differential equations would show us, that there are two new seminvariants which cannot be formed from the previous seven by the Wronskian process. But, while it is possible to determine them, the amount of labor necessary for this is very great, and we prefer to avoid it by introducing from now on a normal form, which we shall consider as being preserved by all transformations of the differential equations (1).

This normal form can in general be obtained by merely solving a quadratic equation. We can make i and i' vanish by choosing ϕ, χ, ψ and ω appropriately. In fact if we put

$$\phi = m + n + \sqrt{(m+n)^2 - 4ii'}, \quad \psi = -2i,$$
$$\chi = m + n - \sqrt{(m+n)^2 - 4ii'}, \quad \omega = -2i,$$

the pseudo-invariants I and I' for the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

will be zero, as shown by equations (22). This can always be done unless

$$(m+n)^2 - 4ii' = t^2 - 4kl = 0,$$

which case we shall leave aside for the present.

Suppose that this reduction has been effected, so that $i = i' = 0$. The transformations (3) do not disturb these relations. But the transformations (21) do,

in general. Equations (22), together with the condition $\phi\omega - \psi\chi \neq 0$, show that the only transformations of form (21), which leave invariant the system of equations $i = i' = 0$, are those for which either

$$\psi = \chi = 0 \quad \text{or} \quad \phi = \omega = 0.$$

But such transformations either convert every pseudo-invariant into itself, or change its sign, or at most interchange two of them, so that

$$k^2,\ l^2,\ t^2, \quad rr',\ r + r', \quad ss',\ s + s'$$

are invariant for all such transformations.

If, however, we disregard those transformations that interchange the two operators Ω and Ω' we may say that *for the normal form of our system of partial differential equations, which is defined by the conditions* $i = i' = 0$, *the quantities*

$$k,\ l,\ t,\ r,\ s,\ r',\ s'$$

and those obtained from them by the Wronskian process constitute a complete system of seminvariants.

This normal form can always be obtained by solving a quadratic, except if

$$t^2 - 4kl = 0.$$

§5.—*Determination of the invariants.*

If new independent variables be introduced by putting

$$\xi_k = \lambda_k(x_1, x_2), \qquad (k = 1, 2),$$

we find

$$\frac{\partial y}{\partial x_k} = \frac{\partial y}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_k} + \frac{\partial y}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_k}, \qquad (k = 1, 2).$$

If then we put

$$\frac{\partial \xi_1}{\partial x_1} = \lambda, \quad \frac{\partial \xi_1}{\partial x_2} = \mu, \quad \frac{\partial \xi_2}{\partial x_1} = \nu, \quad \frac{\partial \xi_2}{\partial x_2} = \rho,$$

we must assume $\lambda\rho - \mu\nu$ different from zero. If we denote the coefficients of the transformed system of differential equations again by $\bar{a}$, $\bar{b}$, etc., we find

$$\left.\begin{aligned}
\bar{a} &= a\lambda + b\mu, & \bar{a}' &= a'\lambda + b'\mu,\\
\bar{b} &= a\nu + b\rho, & \bar{b}' &= a'\nu + b'\rho,\\
\bar{c} &= c\lambda + d\mu, & \bar{c}' &= c'\lambda + d'\mu,\\
\bar{d} &= c\nu + d\rho, & \bar{d}' &= c'\nu + d'\rho,\\
\bar{e} &= e, & \bar{e}' &= e',\\
\bar{f} &= f, & \bar{f}' &= f'.
\end{aligned}\right\} \qquad (27)$$

From these equations it is easy to obtain the following:

$$\left.\begin{aligned}\bar{k} &= \lambda^2 k + \mu^2 l + \lambda\mu t,\\ \bar{l} &= \nu^2 k + \rho^2 l + \nu\rho t,\\ \bar{t} &= 2\lambda\nu k + (\lambda\rho + \mu\nu)\, t + 2\mu\rho l,\end{aligned}\right\} \tag{28}$$

whence

$$t^2 - 4kl \tag{29}$$

is seen to be an invariant.

It is necessary to obtain formulae which shall show how to find the effect of this transformation upon the derivatives of a given function. Let f be such a function of a, b, c, etc., and let $\bar{f}$ denote the corresponding function of $\bar{a}$, $\bar{b}$, etc., these being functions of ξ_1 and ξ_2, and therefore also of x_1 and x_2. We have

$$d\xi_1 = \lambda dx_1 + \mu dx_2, \quad d\xi_2 = \nu dx^1 + \rho dx_2$$

whence

$$\Delta dx_1 = \rho d\xi_1 - \mu d\xi_2, \quad \Delta dx_2 = -\nu d\xi_1 + \lambda d\xi_2,$$

where

$$\Delta = \lambda\rho - \mu\nu.$$

Therefore, we find

$$\Delta \frac{\partial \bar{f}}{\partial \xi_1} = \frac{\partial \bar{f}}{\partial x_1}\rho - \frac{\partial \bar{f}}{\partial x_2}\nu, \quad \Delta \frac{\partial \bar{f}}{\partial \xi_2} = -\frac{\partial \bar{f}}{\partial x_1}\mu + \frac{\partial \bar{f}}{\partial x_2}\lambda. \tag{30}$$

In particular, let the transformation be infinitesimal, so that

$$\lambda = 1 + \varkappa\varepsilon, \quad \mu = \pi\varepsilon, \quad \nu = \sigma\varepsilon, \quad \rho = 1 + \tau\varepsilon, \tag{31}$$

where ε is an infinitesimal, while $\varkappa$, π, σ and τ are arbitrary functions of x_1 and x_2. Then, neglecting infinitesimals of higher order,

$$\bar{f} = f + \delta f, \quad \Delta = 1 + (\varkappa + \tau)\,\varepsilon,$$

and, therefore, substituting in (30),

$$\left.\begin{aligned}\delta\left(\frac{\partial f}{\partial x_1}\right) &= \frac{\partial(\delta f)}{\partial x_1} - \left(\varkappa\frac{\partial f}{\partial x_1} + \sigma\frac{\partial f}{\partial x_2}\right)\varepsilon,\\ \delta\left(\frac{\partial f}{\partial x_2}\right) &= \frac{\partial(\delta f)}{\partial x_2} - \left(\pi\frac{\partial f}{\partial x_1} + \tau\frac{\partial f}{\partial x_2}\right)\varepsilon.\end{aligned}\right\} \tag{32}$$

We proceed to find the infinitesimal transformations of the various seminvariants. We have first (omitting the infinitesimal factor ε),

$$\left.\begin{aligned}\delta a &= \varkappa a + \pi b, & \delta a' &= \varkappa a' + \pi b',\\ \delta b &= \sigma a + \tau b, & \delta b' &= \sigma a' + \tau b',\\ \delta c &= \varkappa c + \pi d, & \delta c' &= \varkappa c' + \pi d',\\ \delta d &= \sigma c + \tau d, & \delta d' &= \sigma c' + \tau d',\\ \delta e &= \delta f = 0, & \delta e' &= \delta f' = 0.\end{aligned}\right\} \tag{33}$$

Substituting (31) in (28), we find

$$\left.\begin{aligned}\delta k &= 2\varkappa k + \pi t,\\ \delta l &= 2\tau l + \sigma t,\\ \delta t &= 2\sigma k + (\varkappa + \tau)\, t + 2\pi l.\end{aligned}\right\} \tag{34}$$

Therefore, making use of (32),

$$\left.\begin{aligned}\delta(k_1, t) &= 2\varkappa(k_1, t) + 2\pi(k_1, l) - \sigma(k_2, t) + \tau(k_1, t)\\ &\qquad + \varkappa_1 kt + \pi_1(t^2 - 2kl) - 2\sigma_1 k^2 - \tau_1 kt,\\ \delta(k_2, t) &= 3\varkappa(k_2, t) + \pi[2(k_2, l) - (k_1, t)]\\ &\qquad + \varkappa_2 kt + \pi_2(t^2 - 2kl) - 2\sigma_2 k^2 - \tau_2 kt,\\ \delta(l_1, t) &= -\sigma[2(k_1, l) + (l_2, t)] + 3\tau(l_1, t)\\ &\qquad + \tau_1 lt + \sigma_1(t^2 - 2kl) - 2\pi_1 l^2 - \varkappa_1 lt,\\ \delta(l_2, t) &= \varkappa(l_2, t) - \pi(l_1, t) - 2\sigma(k_2, l) + 2\tau(l_2, t)\\ &\qquad + \tau_2 lt + \sigma_2(t^2 - 2kl) - 2\pi_2 l^2 - \varkappa_2 lt.\end{aligned}\right\} \tag{35}$$

Similarly, remembering the defining equations (12) of p, q, p', q', we find

$$\left.\begin{aligned}\delta p &= -\varkappa_1 a - \pi_1 b - \sigma_2 a - \tau_2 b, & \delta p' &= -\varkappa_1 a' - \pi_1 b' - \sigma_2 a' - \tau_2 b',\\ \delta q &= -\varkappa_1 c - \pi_1 d - \sigma_2 c - \tau_2 d, & \delta q' &= -\varkappa_1 c' - \pi_1 d' - \sigma_2 c' - \tau_2 d',\end{aligned}\right\} \tag{36}$$

whence

$$\left.\begin{aligned}\delta r &= \varkappa r + \pi s - (\pi_1 + \tau_2)\, i, & \delta r' &= \varkappa r' + \pi s' - (\pi_1 + \tau_2)\, i',\\ \delta s &= \sigma r + \tau s + (\varkappa_1 + \sigma_2)\, i, & \delta s' &= \sigma r' + \tau s' + (\varkappa_1 + \sigma_2)\, i'.\end{aligned}\right\} \tag{37}$$

For the normal form for which $i = i' = 0$, we have

$$\left.\begin{aligned}\delta r &= \varkappa r + \pi s, & \delta s &= \sigma r + \tau s,\\ \delta r' &= \varkappa r' + \pi s', & \delta s' &= \sigma r' + \tau s'.\end{aligned}\right\} \tag{38}$$

Any function of the seminvariants k, l, t; (k_i, t), (l_i, t); r, s, r' and s', which is an absolute invariant under the most general transformation leaving the normal form $i = i' = 0$ unchanged, must therefore be a solution of the system of twelve partial differential equations which is represented by the following table. The letters $\varkappa$, τ, etc., to the left of each row indicate that the quantities in that row are the coefficients of the equation obtained by equating to zero the coefficient of $\varkappa$, τ, etc. in the general expression for δf. The letter at the top of each column indicates that the quantities in that column are the coefficients of the partial derivatives with respect to that quantity.

(39)

	k	l	t	(k^1, t)	(k_2, t)	(l_1, t)	(l_2, t)	r	r'	s	s'
$\varkappa$	$2k$	0	t	$2(k_1, t)$	$3(k_2, t)$	0	(l_2, t)	r	r'	0	0
π	t	0	$2l$	$2(k_1, l)$	$2(k_2, l) - (k_1, t)$	0	$-(l_1, t)$	s	s'	0	0
σ	0	t	$2k$	$-(k_2, t)$	0	$-2(k_1, l) - (l_2, t)$	$-2(k_2, l)$	0	0	r	r'
τ	0	$2l$	t	$+(k_1, t)$	0	$3(l_1, t)$	$2(l_2, t)$	0	0	s	s'
$\varkappa_1$	0	0	0	kt	0	$-lt$	0	0	0	0	0
π_1	0	0	0	$t^2 - 2kl$	0	$-2l^2$	0	0	0	0	0
σ_1	0	0	0	$-2k^2$	0	$t^2 - 2kl$	0	0	0	0	0
τ_1	0	0	0	$-kt$	0	lt	0	0	0	0	0
$\varkappa_2$	0	0	0	0	kt	0	$-lt$	0	0	0	0
π_2	0	0	0	0	$t^2 - 2kl$	0	$-2l^2$	0	0	0	0
σ_2	0	0	0	0	$-2k^2$	0	$t^2 - 2kl$	0	0	0	0
τ_2	0	0	0	0	$-kt$	0	lt	0	0	0	0

The determinant of equations π_2 and σ_2 is $t^2(t^2-4kl)$, and therefore not zero. This shows that (k_2, t) and (l_2, t) cannot occur in such an invariant. From equations π_1 and σ_1 one concludes that the invariants are also independent of (k_1, t) and (l_1, t). The system now reduces to one of four equations in seven variables, so that there must be three absolute or four relative invariants. Of these we already know one, viz.

$$\mathfrak{F} = t^2 - 4kl. \tag{40}$$

The other three are

$$\mathfrak{G} = lr^2 - trs + ks^2, \quad \mathfrak{G}' = lr'^2 - tr's' + ks'^2, \quad \mathfrak{H} = rs' - r's. \tag{41}$$

§6.—*General properties of invariants and covariants.*

We have noticed in the preceding paragraphs that every invariant is multiplied by a power of each of the determinants $\alpha\delta-\beta\gamma$, $\phi\omega-\psi\chi$, $\frac{\partial\xi_1}{\partial x_1}\frac{\partial\xi_2}{\partial x_2}-\frac{\partial\xi_1}{\partial x_2}\frac{\partial\xi_2}{\partial x_1}$, when all of the transformations here considered are performed. This could have been proved a priori without first computing the invariants, and is also true of the covariants. The proof for this would be very much like the corresponding proof in the theory of algebraic invariants.

If we make the special transformation

$$\bar{y} = \alpha y, \quad \bar{z} = \alpha z, \quad \alpha = \text{const.},$$

it is seen that every covariant must be a homogeneous function of y, z, y_k, z_k, etc., as well as of the coefficients a, b, etc., and of their derivatives.

Let ψ be any pseudo-covariant homogeneous of degree k in the coefficients, and of degree i in the variables. If we make the transformation

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z} \tag{1}$$

or

$$\bar{y} = \frac{\delta y - \beta z}{\alpha\delta - \beta\gamma}, \quad \bar{z} = \frac{-\gamma y + \alpha z}{\alpha\delta - \beta\gamma},$$

we find from equations (6)

$$\bar{a} = a\alpha + c\gamma, \text{ etc.},$$

and since we know that $\bar{\psi}$ will be equal to ψ multiplied by a power of $\alpha\delta-\beta\gamma$, the exponent of this power must be $\frac{k-i}{2}$ since $\bar{\psi}$ will be a homogeneous function of $\alpha, \beta, \gamma, \delta$ of degree $k-i$ while $\alpha\delta-\beta\gamma$ is homogeneous of degree 2.

For our purposes of calculation it was convenient to write our transformation in the form (1). But if we make it a rule to always write our equations of transformation solved with respect to the new variables, we should have

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z,$$

and correspondingly

$$\bar{\psi} = (\alpha\delta - \beta\gamma)^{\frac{i-k}{2}} \psi.$$

If ψ is also a semicovariant, we find that for the transformations, which correspond to replacing the system $\Omega = \Omega' = 0$ by

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

ψ is multiplied by

$$(\phi\omega - \psi\chi)^{\frac{k}{2}}.$$

Make now a particular transformation of the independent variables

$$\xi_1 = cx_1, \quad \xi_2 = cx_2,$$

where c is a constant. Then

$$\bar{y} = y, \qquad \bar{y}_k = c^{-1} y_k, \quad \bar{y}_{kl} = c^{-2} y_{kl}, \text{ etc.},$$
$$\bar{a} = ca, \text{ etc.}, \quad \bar{e} = e, \text{ etc.}$$

Every quantity is multiplied by some power of c. Let us say that a quantity multiplied by c^w is of weight w. Then clearly every invariant or covariant must be isobaric, and an arbitrary transformation of the dependent variables multiplies it by $(\lambda\rho - \mu\nu)^{\frac{w}{2}}$.

Let us unite our results in a theorem.

Every rational function of y, z, a, b, c, etc., which has the property of being absolutely invariant is a quotient of two homogeneous isobaric rational integral functions of these quantities which are relative invariants.

Let $I_{i,k}^{(w)}$ be such an integral rational invariant function, of weight w, of degree i in the variables, and of degree k in the coefficients. If the system

$$\Omega = \Omega' = 0$$

be transformed by putting first

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z, \quad \xi_1 = f_1(x_1, x_2), \quad \xi_2 = f_2(x_1, x_2),$$

and then by replacing it by

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

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Every rational function of y, z, a, b, c, etc., which has the property of being absolutely invariant is a quotient of two homogeneous isobaric rational integral functions of these quantities which are relative invariants.

Let $I_{i,k}^{(w)}$ be such an integral rational invariant function, of weight w, of degree i in the variables, and of degree k in the coefficients. If the system

$$\Omega = \Omega' = 0$$

be transformed by putting first

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z, \quad \xi_1 = f_1(x_1, x_2), \quad \xi_2 = f_2(x_1, x_2),$$

and then by replacing it by

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

the transformed invariant is

$$\overline{I}_{ik}^{(w)} = (\alpha\delta - \beta\gamma)^{\frac{i-k}{2}} (\phi\omega - \psi\chi)^{\frac{k}{2}} (\lambda\rho - \mu\nu)^{\frac{w}{2}} I_{ik}^{(w)}.$$

Since the exponents must be integers if I is a rational invariant, we see further, that *both of the degrees and the weight of any invariant function must be even numbers.*

For invariants proper $i = 0$. From two such invariants of degrees k_1, k_2 of weights w_1, w_2 we can obtain an absolute invariant, if and only if two integers λ_1 and λ_2 can be determined so that

$$\lambda_1 k_1 + \lambda_2 k_2 = 0,$$
$$\lambda_1 w_1 + \lambda_2 w_2 = 0,$$

i. e. if, and only if,

$$k_1 w_2 - k_2 w_1 = 0.$$

For our four invariants we have the following table of weight and degrees:

Invariant.	Weight.	Degree.
$\mathfrak{F}$	4	4
$\mathfrak{H}$	2	4
$\mathfrak{G}$	4	6
$\mathfrak{G}'$	4	6

Therefore only one absolute invariant is obtained namely $\mathfrak{G} / \mathfrak{G}'$.

§7.—*Geometrical interpretation.*

Let us write the equations (1), putting into evidence the dependent variables, as follows:

$$\Omega(y, z) = 0, \quad \Omega'(y, z) = 0, \tag{42}$$

and let $y^{(k)}$, $z^{(k)}$ for $k = 1, 2, 3, 4$ be four pairs of simultaneous solutions, so that

$$\Omega(y^{(k)}, z^{(k)}) = 0, \quad \Omega'(y^{(k)}, z^{(k)}) = 0, \qquad (k = 1, 2, 3, 4). \tag{43}$$

Let us interpret $y^{(k)}$ and $z^{(k)}$ as the homogeneous coordinates of two points P_y and P_z in space. We shall have

$$y^{(k)} = f^{(k)}(x_1, x_2) \quad z^{(k)} = g^{(k)}(x_1, x_2) \qquad (k = 1, 2, 3, 4) \tag{44}$$

when these solutions are known. As x_1 and x_2 assume all of the values of which they are capable, P_y and P_z will describe two surfaces S_y and S_z, which

moreover will be placed in a point-to-point correspondence, those points being corresponding points which belong to the same set of values (x_1, x_2).

An arbitrary change of the independent variables, clearly does not change these surfaces, nor does it change the point-to-point correspondence between them. The most general transformation of the form

$$y = \alpha(x_1, x_2)\eta + \beta(x_1, x_2)\zeta, \quad z = \gamma(x_1, x_2)\eta + \delta(x_1, x_2)\zeta,$$

clearly converts the points P_y and P_z into two others, P_η and P_ζ, situated on the line L_{yz} joining P_y and P_z. If then we consider the congruence of lines joining the corresponding points of the two surfaces S_y and S_z, this congruence will be the same for all of the systems of differential equations equivalent to (1) by transformations of the infinite group considered in this paper.

But it is necessary to examine this somewhat more closely, so that we may see to what extent this geometrical image is characteristic of our system of differential equations.

Let the functions $f^{(k)}$ and $g^{(k)}$ in (44) be any functions of x_1 and x_2 whatever, and put

$$\left.\begin{aligned} y &= c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)} + c_4 y^{(4)}, \\ z &= c_1 z^{(1)} + c_2 z^{(2)} + c_3 z^{(3)} + c_4 z^{(4)}, \end{aligned}\right\} \tag{45}$$

where $c_1 \ldots\ldots c_4$ are arbitrary constants. By differentiation we find the following four equations:

$$\left.\begin{aligned} \frac{\partial y}{\partial x_1} &= \sum_{k=1}^{4} c_k \frac{\partial y^{(k)}}{\partial x_1}, & \frac{\partial y}{\partial x_2} &= \sum_{k=1}^{4} c_k \frac{\partial y^{(k)}}{\partial x_2}, \\ \frac{\partial z}{\partial x_1} &= \sum_{k=1}^{4} c_k \frac{\partial z^{(k)}}{\partial x_1}, & \frac{\partial z}{\partial x_2} &= \sum_{k=1}^{4} c_k \frac{\partial z^{(k)}}{\partial x_2}. \end{aligned}\right\} \tag{46}$$

We can eliminate $c_1, \ldots\ldots c_4$ in two different ways, and thus obtain the two equations:

$$D_1 = \begin{vmatrix} y, & y^{(1)}, & \ldots\ldots & y^{(4)} \\ \frac{\partial y}{\partial x_1}, & \frac{\partial y^{(1)}}{\partial x_1}, & \ldots\ldots & \frac{\partial y^{(4)}}{\partial x_1} \\ \frac{\partial y}{\partial x_2}, & \frac{\partial y^{(1)}}{\partial x_2}, & \ldots\ldots & \frac{\partial y^{(4)}}{\partial x_2} \\ \frac{\partial z}{\partial x_1}, & \frac{\partial z^{(1)}}{\partial x_1}, & \ldots\ldots & \frac{\partial z^{(4)}}{\partial x_1} \\ \frac{\partial z}{\partial x_2}, & \frac{\partial z^{(1)}}{\partial x_2}, & \ldots\ldots & \frac{\partial z^{(4)}}{\partial x_2} \end{vmatrix} = 0, \quad D_2 = \begin{vmatrix} z, & z^{(1)}, & \ldots\ldots & z^{(4)} \\ \frac{\partial y}{\partial x_1}, & \frac{\partial y^{(1)}}{\partial x_1}, & \ldots\ldots & \frac{\partial y^{(4)}}{\partial x_1} \\ \frac{\partial y}{\partial x_2}, & \frac{\partial y^{(1)}}{\partial x_2}, & \ldots\ldots & \frac{\partial y^{(4)}}{\partial x_2} \\ \frac{\partial z}{\partial x_1}, & \frac{\partial z^{(1)}}{\partial x_1}, & \ldots\ldots & \frac{\partial z^{(4)}}{\partial x_1} \\ \frac{\partial z}{\partial x_2}, & \frac{\partial z^{(1)}}{\partial x_2}, & \ldots\ldots & \frac{\partial z^{(4)}}{\partial x_2} \end{vmatrix} = 0, \tag{47}$$

which are of the form of system (1) and are satisfied by the four pairs of functions $y^{(k)}$, $z^{(k)}$. Moreover, this system is essentially unique. We have eliminated in such a manner as to make $f=0$, $e'=0$. Had we eliminated differently, the reduction to this form could be accomplished *a posteriori*, and, moreover, as equations (21) show, in essentially one way.

In order that (47) may be a determinate system, it is, of course, necessary that each of the matrices

$$\left.\begin{array}{l} \left\| y^{(k)}, \quad \dfrac{\partial y^{(k)}}{\partial x_1}, \quad \dfrac{\partial y^{(k)}}{\partial x_2}, \quad \dfrac{\partial z^{(k)}}{\partial x_1}, \quad \dfrac{\partial z^{(k)}}{\partial x_2} \right\| \\ \left\| z^{(k)}, \quad \dfrac{\partial y^{(k)}}{\partial x_1}, \quad \dfrac{\partial y^{(k)}}{\partial x_2}, \quad \dfrac{\partial z^{(k)}}{\partial x_1}, \quad \dfrac{\partial z^{(k)}}{\partial x_2} \right\| \end{array}\right\}, \qquad (k=1,2,3,4) \tag{48}$$

should contain at least one non-vanishing determinant of the fourth order. This we shall assume to be the case, and we will not here enter upon a discussion of the meaning of the exceptional cases.

We see that to every congruence there belongs, in general, a system of partial differential equations of the form (1), namely, the system (47).

But there are an infinite number of congruences giving rise to the same system of differential equations. In the first place it is clear that (for arbitrary constant coefficients c_k)

$$y=\sum_{k=1}^{4} c_k y^{(k)}, \quad z=\sum_{k=1}^{4} c_k z^{(k)}$$

also constitute a simultaneous system of solutions. But this means that every congruence, obtained from the given one by projective transformation, gives rise to the same system of differential equations. But more than that. Equations (47) merely express the fact that, if we denote by Y and Z any possible pair of solutions, while $y^{(k)}$, $z^{(k)}$ represent four particular pairs of solutions, the following conditions must be satisfied

$$\left.\begin{array}{l} Y=\displaystyle\sum_{k=1}^{4}\phi_k y^{(k)}, \qquad Z=\sum_{k=1}^{4}\phi_k z^{(k)}, \\ \dfrac{\partial Y}{\partial x_\lambda}=\displaystyle\sum_{k=1}^{4}\phi_k\frac{\partial y^{(k)}}{\partial x_\lambda}, \quad \frac{\partial Z}{\partial x_\lambda}=\sum_{k=1}^{4}\phi_k\frac{\partial z^{(k)}}{\partial x_\lambda}, \quad (\lambda=1,2). \end{array}\right\} \tag{49}$$

If the quantities ϕ_k are constants we get the case just noticed. But the equations

(49) may be satisfied even if the quantities ϕ_k are not constants. All that is necessary for this, is that they satisfy the conditions

$$\sum_{k=1}^{4} y^{(k)} \frac{\partial \phi_k}{\partial x_\lambda} = 0, \quad \sum_{k=1}^{4} z^{(k)} \frac{\partial \phi_k}{\partial x_\lambda} = 0, \qquad (\lambda = 1, 2). \tag{50}$$

But this is equivalent to demanding that

$$\frac{\partial \phi_k}{\partial x_1} \quad \text{and} \quad \frac{\partial \phi_k}{\partial x_2}$$

shall be the coordinates of two planes passing through L_{yz}.

If we denote the coordinates of any two such planes by $u^{(k)}$ and $v^{(k)}$, we may put

$$\frac{\partial \phi_k}{\partial x_1} = \lambda_1 u^{(k)} + \mu_1 v^{(k)}, \quad \frac{\partial \phi_k}{\partial x_2} = \lambda_2 u^{(k)} + \mu_2 v^{(k)}, \tag{51}$$

where $\lambda_1, \mu_1, \lambda_2, \mu_2$ are arbitrary functions subject merely to the integrability conditions

$$\frac{\partial}{\partial x_2}(\lambda_1 u^{(k)} + \mu_1 v^{(k)}) = \frac{\partial}{\partial x_1}(\lambda_2 u^{(k)} + \mu_2 v^{(k)}). \qquad (k = 1, 2, 3, 4). \tag{52}$$

We can express this result in another way. If $\lambda^{(1)} \ldots \lambda^{(4)}, \mu^{(1)} \ldots \mu^{(4)}$ are eight arbitrary functions, and we denote the minors of $t^{(1)} \ldots t^{(4)}$ in the two determinants

$$\begin{vmatrix} t^{(1)} & t^{(2)} & t^{(3)} & t^{(4)} \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ \lambda^{(1)} & \lambda^{(2)} & \lambda^{(3)} & \lambda^{(4)} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} t^{(1)} & t^{(2)} & t^{(3)} & t^{(4)} \\ y^{(1)} & \ldots & \ldots & y^{(4)} \\ z^{(1)} & \ldots & \ldots & z^{(4)} \\ \mu^{(1)} & \ldots & \ldots & \mu^{(4)} \end{vmatrix}$$

by $u^{(1)} \ldots u^{(4)}$ and $v^{(1)} \ldots v^{(4)}$ respectively, then $u^{(k)}$ and $v^{(k)}$ are the coordinates of any two planes passing through the line L_{yz} of the congruence. Determine the functions $\lambda^{(1)} \ldots \mu^{(4)}$ in the most general way, so as to satisfy the conditions

$$\frac{\partial u^{(k)}}{\partial x_2} - \frac{\partial v^{(k)}}{\partial x_1} = 0,$$

so that we may put

$$u^{(k)} = \frac{\partial \phi_k}{\partial x_1}, \quad v^{(k)} = \frac{\partial \phi_k}{\partial x_2},$$

Then the most general solution of system (1) is

$$Y = \sum_{k=1}^{4} \phi_k y^{(k)}, \quad Z = \sum_{k=1}^{4} \phi_k z^{(k)},$$

the four pairs of particular solutions $(y^{(k)}, z^{(k)})$ being such that not all of the determinants of the fourth order in the matrices (48) become equal to zero.

Under these assumptions we can deduce a system of partial differential equations for $u^{(k)}$ and $v^{(k)}$ similar in form to (1), and which may be spoken of as being adjoined to (1). For, with our interpretation, it represents the same congruence as (1), the lines of the congruence, however, being obtained as intersections of corresponding planes instead of as lines joining corresponding points.

As a matter of fact we shall have

$$\left.\begin{array}{ll}
\sum u^{(k)} y^{(k)} = 0, & \sum u^{(k)} z^{(k)} = 0 \\
\sum \left(u^{(k)} \frac{\partial y^{(k)}}{\partial x_1} + \frac{\partial u^{(k)}}{\partial x_1} y^{(k)} \right) = 0, & \cdots\cdots\cdots \\
\sum \left(u^{(k)} \frac{\partial y^{(k)}}{\partial x_2} + \frac{\partial u^{(k)}}{\partial x_2} y^{(k)} \right) = 0, & \cdots\cdots\cdots \\
\quad \text{etc.} &
\end{array}\right\} \tag{53}$$

Now we have $\quad \Omega(y^{(k)}, z^{(k)}) = 0, \quad \Omega'(y^{(k)}, z^{(k)}) = 0$

and therefore $\quad \sum u^{(k)} \Omega(y^{(k)}, z^{(k)}) = 0, \quad$ etc.

whence, making use of (53),

$$a \sum u^{(k)} \frac{\partial y^{(k)}}{\partial x_1} + b \sum u^{(k)} \frac{\partial y^{(k)}}{\partial x_2} + c \sum u^{(k)} \frac{\partial z^{(k)}}{\partial x_1} + d \sum u^{(k)} \frac{\partial z^{(k)}}{\partial x_2} = 0.$$

or again according to (53),

$$a \sum y^{(k)} \frac{\partial u^{(k)}}{\partial x_1} + b \sum y^{(k)} \frac{\partial u^{(k)}}{\partial x_2} + c \sum z^{(k)} \frac{\partial u^{(k)}}{\partial x_1} + d \sum z^{(k)} \frac{\partial u^{(k)}}{\partial x_2} = 0.$$

We find thus the first of the following four equations:

$$\sum y^{(k)} \left(a \frac{\partial u^{(k)}}{\partial x_1} + b \frac{\partial u^{(k)}}{\partial x_2} \right) + \sum z^{(k)} \left(c \frac{\partial u^{(k)}}{\partial x_1} + d \frac{\partial u^{(k)}}{\partial x_2} \right) = 0,$$

$$\sum y^{(k)} \left(a \frac{\partial v^{(k)}}{\partial x_1} + b \frac{\partial v^{(k)}}{\partial x_2} \right) + \sum z^{(k)} \left(c \frac{\partial v^{(k)}}{\partial x_1} + d \frac{\partial v^{(k)}}{\partial x_2} \right) = 0,$$

$$\sum y^{(k)} \left(a' \frac{\partial u^{(k)}}{\partial x_1} + b' \frac{\partial u^{(k)}}{\partial x_2} \right) + \sum z^{(k)} \left(c' \frac{\partial u^{(k)}}{\partial x_1} + d' \frac{\partial u^{(k)}}{\partial x_2} \right) = 0,$$

$$\sum y^{(k)} \left(a' \frac{\partial v^{(k)}}{\partial x_1} + b' \frac{\partial v^{(k)}}{\partial x_2} \right) + \sum z^{(k)} \left(c' \frac{\partial v^{(k)}}{\partial x_1} + d' \frac{\partial v^{(k)}}{\partial x_2} \right) = 0,$$

whence

$$(ab'-a'b)\sum y^{(k)}\frac{\partial u^{(k)}}{\partial x_2}+(ac'-a'c)\sum z^{(k)}\frac{\partial u^{(k)}}{\partial x_1}+(ad'-a'd)\sum z^{(k)}\frac{\partial u^{(k)}}{\partial x_2}=0,$$

$$(ab'-a'b)\sum y^{(k)}\frac{\partial v^{(k)}}{\partial x_1}+(cb'-c'b)\sum z^{(k)}\frac{\partial v^{(k)}}{\partial x_1}+(db'-d'b)\sum z^{(k)}\frac{\partial v^{(k)}}{\partial x_2}=0.$$

Remembering that we have assumed

$$\frac{\partial u^{(k)}}{\partial x_2}=\frac{\partial v^{(k)}}{\partial x_1},$$

we find by subtraction

$$\sum_{k=1}^{4} z^{(k)}\left[(ac'-a'c)\frac{\partial u^{(k)}}{\partial x_1}+(ad'-a'd)\frac{\partial u^{(k)}}{\partial x_2}+(bc'-b'c)\frac{\partial v^{(k)}}{\partial x_1}\right.$$
$$\left.+(bd'-b'd)\frac{\partial v^{(k)}}{\partial x_2}\right]=0.$$

By a similar process we can obtain another equation precisely the same except that $z^{(k)}$ is replaced by $y^{(k)}$. But any set of four quantities which satisfies two relations of the form

$$\sum y^{(k)}w^{(k)}=0,\quad \sum z^{(k)}w^{(k)}=0,$$

may be considered as the coordinates of a plane passing through L_{yz}. We may therefore write down the result that the quantities $u^{(k)}$, $v^{(k)}$ of our previous theorem satisfy a system of partial differential equations of the form

$$(ac'-a'c)\frac{\partial u}{\partial x_1}+(ad'-a'd)\frac{\partial u}{\partial x_2}+(bc'-b'c)\frac{\partial v}{\partial x_1}+(bd'-b'd)\frac{\partial v}{\partial x_2}$$
$$+\nu u+\rho v=0,\quad \frac{\partial u}{\partial x_2}-\frac{\partial v}{\partial x_1}=0\,. \qquad (54)$$

The values of ν and ρ will depend upon the choice of the quantities $\lambda^{(1)}\ldots\ldots\lambda^{(4)}$, $\mu^{(1)}\ldots\ldots\mu^{(4)}$ subject to the conditions of the previous theorem.

It follows at once that the functions ϕ_k are solutions of the differential equation of the second order

$$k\frac{\partial^2\phi}{\partial x_1^2}+t\frac{\partial^2\phi}{\partial x_1\partial x_2}+l\frac{\partial^2\phi}{\partial x_2^2}+\nu\frac{\partial\phi}{\partial x_1}+\rho\frac{\partial\phi}{\partial x_2}=0\,, \tag{55}$$

if we remember the significance of the quantities k, l and t.

§8.—*Developable surfaces of the congruence.*

Suppose that we have four pairs of simultaneous solutions of system (1),

$$y^{(k)}=f^{(k)}(x_1,\,x_2),\quad z^{(k)}=g^{(k)}(x_1,\,x_2),\qquad (k=1,\,2,\,3,\,4) \tag{56}$$

the conditions being fulfilled under which they determine a congruence. If we put in (56),

$$x_1=\phi_1(t),\quad x_2=\phi_2(t), \tag{57}$$

we obtain two curves, c_y and c_z and those lines of the congruence which join corresponding points of the two curves are to be considered. They form a ruled surface of the congruence. The question arises as to the existence of developable surfaces made up of lines of the congruence.

If we imagine (57) substituted in (56), we find for every index k,

$$\frac{dy}{dt}=\frac{\partial y}{\partial x_1}\phi_1'+\frac{\partial y}{\partial x_2}\phi_2',\quad \frac{dz}{dt}=\frac{\partial z}{\partial x_1}\phi_1'+\frac{\partial z}{\partial x_2}\phi_2'.$$

But the surface in question is a developable surface if, and only if, the tangents of the two curves c_y and c_z constructed at corresponding points intersect, i. e. if, and only if, it is possible to satisfy the equation

$$\lambda y+\nu\frac{dy}{dt}+\mu z+\rho\frac{dz}{dt}=0\,,$$

where λ, μ, ν, ρ are functions of t, by putting $y=y^{(k)}$, $z=z^{(k)}$, for ($k=1$, 2, 3, 4), i. e. if, and only if, λ, μ, ν, ρ can be determined so that

$$\lambda y+\mu z+\nu\left(\frac{\partial y}{\partial x_1}\phi_1'+\frac{\partial y}{\partial x_2}\phi_2'\right)+\rho\left(\frac{\partial z}{\partial x_1}\phi_1'+\frac{\partial z}{\partial x_2}\phi_2'\right)=0\,. \tag{58}$$

But y and z satisfy the equations

$$\Omega(y, z) = 0, \quad \Omega'(y, z) = 0.$$

We may combine these into an equation

$$\alpha\Omega(y, z) + \alpha'\Omega'(y, z) = 0$$

which becomes identical with (58) after substituting into it the values (57) for x_1 and x_2.

For the identity of these two equations we must have

$$\left.\begin{aligned} \alpha a + \alpha' a' &= \nu\phi_1', & \alpha c + \alpha' c' &= \rho\phi_1', & \alpha e + \alpha' e' &= \lambda, \\ \alpha b + \alpha' b' &= \nu\phi_2', & \alpha d + \alpha' d' &= \rho\phi_2', & \alpha f + \alpha' f' &= \mu. \end{aligned}\right\} \tag{59}$$

From the first four equations, eliminating α, α', ν and ρ, we obtain

$$\begin{vmatrix} a & a' & \phi_1' & 0 \\ b & b' & \phi_2' & 0 \\ c & c' & 0 & \phi_1' \\ d & d' & 0 & \phi_2' \end{vmatrix} = 0,$$

or, expanding and remembering the significance of k, l and t,

$$l\phi_1'^2 - t\phi_1'\phi_2' + k\phi_2'^2 = 0, \tag{60}$$

a quadratic differential equation, perhaps more clearly written

$$l dx_1^2 - t dx_1 dx_2 + k dx_2^2 = 0. \tag{60a}$$

If this equation be satisfied by choosing x_2 as a function of x_1 in accordance with it, or more symmetrically x_1 and x_2 as functions of t, in accordance with (60), equations (59) can be satisfied. It is possible, therefore, to assemble the lines of the congruence into developable surfaces in an infinity of ways. *The congruence contains two families of ∞^1 developable surfaces, which are obtained by integrating the two differential equations obtained by factoring* (60).

Suppose that $k = l = 0$, $t \neq 0$. Then the two families of developables are obtained by putting either x_1 or x_2 equal to an arbitrary constant, i. e. the curves $x_1 =$ const. on the surfaces S_y and S_z are the curves in which the developables of one family intersect these surfaces, and the curves $x_2 =$ const. are the curves in which

the developables of the other family intersect S_y and S_z. But equations (28) show that we may always choose new independent variables ξ_1 and ξ_2, so that for these $\bar{k}$ and $\bar{l}$ both vanish, if k and l are not zero in the first place, provided $t^2 - 4kl$ is not zero. We have merely to take for ξ_1 and ξ_2 two independent solutions of the equation

$$k\left(\frac{\partial\xi}{\partial x_1}\right)^2 + t\frac{\partial\xi}{\partial x_1}\frac{\partial\xi}{\partial x_2} + l\left(\frac{\partial\xi}{\partial x_2}\right)^2 = 0\,.$$

Therefore, if $t^2 - 4kl$ does not vanish, and we introduce new independent variables ξ_1 and ξ_2 by putting

$$\xi_1 = h_1(x_1, x_2), \quad \xi_2 = h_2(x_1, x_2),$$

where ξ_1 and ξ_2 are two independent solutions of the differential equation

$$k\left(\frac{\partial\xi}{\partial x_1}\right)^2 + t\frac{\partial\xi}{\partial x_1}\frac{\partial\xi}{\partial x_2} + l\left(\frac{\partial\xi}{\partial x_2}\right)^2 = 0\,, \tag{61}$$

the curves $\xi_1 = const.$ and $\xi_2 = const.$ represent the curves on both surfaces S_y and S_z in which these surfaces intersect the two families of developable surfaces of the congruence. Moreover this choice of parameters is characterized by the conditions $k = l = 0$.

It must be possible to determine the functions $\alpha, \beta, \gamma, \delta$ in the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta \tag{62}$$

in such a way that the surfaces S_η and S_ζ may become those two surfaces which contain the edges of regression of the two families of developable surfaces. Suppose that the surface S_η is the locus of the edges of regression of that family of developables corresponding to $\xi_1 = $ const. and S_ζ that of the developables corresponding to $\xi_2 =$ const. Then P_ζ must lie on a tangent to the curve $\xi_1 =$ const. which passes through P_η, i. e. an equation of the form

$$\bar{b}\frac{\partial\eta}{\partial\xi_2} + \bar{e}\eta + \bar{f}\zeta = 0, \tag{63a}$$

must be satisfied, and similarly an equation of the form

$$\bar{c}' \frac{\partial \zeta}{\partial \xi_1} + \bar{e}' \eta + \bar{f}' \zeta = 0. \tag{63b}$$

Moreover, we have already assumed that the preliminary reduction

$$k = l = 0$$

has been made. We must therefore have

$$\bar{a} = 0, \quad \bar{c} = \bar{d} = 0, \quad \bar{a}' = \bar{b}' = 0, \quad \bar{d}' = 0, \quad \bar{k} = \bar{l} = 0,$$

where $\bar{a}$, etc., are expressed in terms of a, b, c, etc., by equations (6).

The conditions $\bar{c} = \bar{d} = 0$ and $\bar{a}' = \bar{b}' = 0$ give $i = i' = 0$ respectively, for neither β and δ nor α and γ can vanish simultaneously, as that would make $\alpha\delta - \beta\gamma$ vanish. We can therefore put

$$\beta = -d, \quad \delta = b, \quad \alpha = -c', \quad \gamma = a',$$

and then find

$$\bar{a} = -(ac' - a'c) = -k = 0, \quad \bar{d}' = bd' - b'd = l = 0,$$

so that all of the conditions are satisfied.

The lines of the congruences are clearly the common tangents of the two surfaces S_η and S_ζ thus determined. These are usually looked upon as two sheets of a single surface, the *focal surface* of the congruence.

If, then, for a system of form (1) the conditions $k = l = i = i' = 0$ are satisfied, the two sheets of the focal surface of the congruence may be obtained by making the transformation

$$y = c'\eta + d\zeta, \quad z = a'\eta + b\zeta.$$

If t also vanishes, they coincide. The curves $x_1 =$ const. on S_η and the curves $x_2 =$ const. on S_ζ are the cuspidal edges of the two families of developables which are contained in the congruence.

We can, however, obtain the focal surface in another, more elegant manner.

Let the system (1) or

$$\Omega = 0, \quad \Omega' = 0$$

be given in the first place, and let its coefficients be a, b, c, etc. Consider in place of this the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0,$$

whose coefficients A, B, C, etc., are given by (21). Then transform this system by putting

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

and denote the new coefficients by $\overline{A}$, $\overline{B}$, $\overline{C}$, etc. We shall then have, in particular

$$\left.\begin{aligned} \overline{C} &= (a\beta + c\delta)\,\phi + (a'\beta + c'\delta)\,\psi, & \overline{A}' &= (a\alpha + c\gamma)\,\chi + (a'\alpha + c'\gamma)\,\omega, \\ \overline{D} &= (b\beta + d\delta)\,\phi + (b'\beta + d'\delta)\,\psi, & \overline{B}' &= (b\alpha + d\gamma)\,\chi + (b'\alpha + d'\gamma)\,\omega. \end{aligned}\right\} \quad (64)$$

Let us determine these transformations, so that

$$\overline{C} = \overline{D} = \overline{A}' = \overline{B}' = 0. \quad (65)$$

Then the new system has the form

$$\left.\begin{aligned} \overline{A}\frac{\partial\eta}{\partial x_1} + \overline{B}\frac{\partial\eta}{\partial x_2} + \overline{E}\eta + \overline{F}\zeta = 0, \\ \overline{C}'\frac{\partial\zeta}{\partial x_1} + \overline{D}'\frac{\partial\zeta}{\partial x_2} + \overline{E}'\eta + \overline{F}'\zeta = 0. \end{aligned}\right\} \quad (66)$$

The first equation shows that P_ζ is in the plane tangent to S_η at P_η, and the second shows that P_η is in the plane tangent to S_ζ at P_ζ. In other words, the line $P_\eta P_\zeta$ is tangent to both S_η and S_ζ, which must therefore be the two sheets of the focal surface.

But in order that equations (65) may be satisfied it is necessary and sufficient, (since the possibilities $\phi = \psi = 0$, and $\chi = \omega = 0$ are excluded, and since the determinant $\alpha\delta - \beta\gamma$ must not vanish) that the ratios $\frac{\beta}{\delta}$ and $\frac{\alpha}{\gamma}$ should be distinct roots of the quadratic

$$(ab' - a'b)\,\lambda^2 + (ad' - bc' + cb' - da')\,\lambda + cd' - c'd = 0,$$

whose discriminant is $t^2 - 4kl$.

But we have

$$(\alpha\delta-\beta\gamma)\,\eta=\delta y-\beta z,\quad (\alpha\delta-\beta\gamma)\,\zeta=-\gamma y+\delta z,$$

whence

$$(\alpha\delta-\beta\gamma)^2\eta\zeta=\gamma\delta\left[-y^2-\frac{\alpha}{\gamma}\;\frac{\beta}{\delta}z^2+\left(\frac{\alpha}{\gamma}+\frac{\beta}{\delta}\right)yz\right]$$

$$=\frac{-\gamma\delta}{ab'-a'b}\left[(ab'-a'b)\,y^2+(cd'-c'd)\,z^2+(ad'-bc'+cb'-da')\,yz\right].$$

We have then the following result. *The factors of the expression*

$$C=(ab'-a'b)\,y^2+(ad'-a'd+cb'-c'b)\,yz+(cd'-c'd)\,z^2 \tag{67}$$

are distinct if $t^2-4kl\neq 0$. *If they are put equal to* η *and* ζ *respectively, the surfaces* S_η *and* S_ζ *are the two sheets of the focal surface of the congruence.*

One may conclude herefrom that the above expression is a *covariant* of our system. This may be verified analytically.

Moreover we learn that the condition

$$t^2-4kl=0$$

means that the two sheets of the focal surface of the congruence coincide.

If this reduction be combined with that of our previous theorem, we notice that every system of form (1) can be reduced to the form

$$\left.\begin{aligned} y_2&=\lambda y+\mu z,\\ z_1&=\nu y+\rho z,\end{aligned}\right\} \tag{68}$$

provided that $t^2-4kl\neq 0$. In that case the surfaces S_y and S_z are the two (distinct) sheets of the focal surface. Moreover, the curves $x_1=$ const. on S_y and the curves $x_2=$ const. on S_z are the cuspidal edges of the two families of developable surfaces of the congruence.

The curves $x_1=$ const. and $x_2=$ const. are conjugate curves on each of these surfaces. For, if we differentiate both members of the first equation of

(68) with respect to x_1 and those of the second with respect to x_2, we find

$$\left.\begin{aligned}\mu y_{12} &= \lambda\mu y_1 + (\mu_1 + \mu\rho)\, y_2 + [\mu(\lambda_1 + \mu\nu) - \lambda(\mu_1 + \mu\rho)]\, y,\\ \nu z_{12} &= (\nu_2 + \lambda\nu)\, z_1 + \nu\rho z_2 \quad + [\nu(\rho_2 + \mu\nu) - \rho(\nu_2 + \lambda\nu)]\, z.\end{aligned}\right\} \tag{69}$$

But any four functions of x_1, x_2 which satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial x_1\,\partial x_2} + a\frac{\partial\theta}{\partial x_1} + b\frac{\partial\theta}{\partial x_2} + c\theta = 0$$

can be taken as the four homogeneous coordinates of the points of a surface on which the curves $x_1 =$ const. and $x_2 =$ const. are conjugate lines.*

If the sheet of the focal surface S_η degenerates into a curve, it must be possible to find a function of x_1 and x_2, such that the ratios of $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$, $\eta^{(4)}$ are functions of θ alone. This is the case if in equations (66), $\bar{F} = 0$. For let $\eta^{(i)}$, $\eta^{(k)}$ be any two solutions of the equation

$$\bar{A}\frac{\partial\eta}{\partial x_1} + \bar{B}\frac{\partial\eta}{\partial x_2} + \bar{E}\eta = 0,$$

then their quotient satisfies the equation

$$\bar{A}\frac{\partial}{\partial x_1}\frac{\eta^{(i)}}{\eta^{(k)}} + \bar{B}\frac{\partial}{\partial x_2}\frac{\eta^{(i)}}{\eta^{(k)}} = 0,$$

whose most general solution is an arbitrary function of any particular solution θ. Moreover any four functions whose ratios are functions of θ alone always satisfy an equation of this form, so that the condition $\bar{F} = 0$ is necessary as well as sufficient.

To express this condition in invariant form, let us assume in the first place that $i = 0$. We have seen that we can always assume that this is the case by choosing the functions ϕ, ψ, χ and ω properly and considering the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0,$$

* Darboux, Théorie des Surfaces, Tome I, p. 122.

instead of $\Omega = \Omega' = 0$. Then make the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta.$$

The first equation should assume the form

$$\bar{a}\frac{\partial\eta}{\partial x_1} + \bar{b}\frac{\partial\eta}{\partial x_2} + \bar{e}\eta = 0.$$

Therefore, we must have $\bar{c} = \bar{d} = \bar{f} = 0$, or

$$\left.\begin{array}{l} a\beta + c\delta = 0, \quad b\beta + d\delta = 0, \\ a\beta_1 + b\beta_2 + c\delta_1 + d\delta_2 + e\beta + f\delta = 0. \end{array}\right\} \tag{70}$$

The first two conditions can be fulfilled by putting

$$\beta : \delta = -c : a = -d : b,$$

since

$$i = ad - bc = 0.$$

If we differentiate each of these conditions with respect to x_1 and x_2 and use the resulting equations to eliminate β_1, β_2, δ_1, δ_2 from the third condition, we find

$$(e - a_1 - b_2)\,\beta + (f - c_1 - d_2)\,\delta = 0,$$

or, according to (12),

$$p\beta + q\delta = 0, \tag{71}$$

which, together with the first two conditions, gives

$$r = aq - cp = 0, \quad s = bq - dp = 0, \tag{72}$$

for β and δ cannot vanish simultaneously. On the other hand, if these conditions are satisfied, one finds $\bar{c} = \bar{d} = \bar{f} = 0$ on making the above transformation.

Therefore, the conditions $r = 0$, $s = 0$, are necessary and sufficient to make one of the sheets of the focal surface degenerate if $i = 0$.

If we refer everything to the normal form for which $i = i' = 0$, which always exists if $t^2 - 4kl \neq 0$, we see that *if* $r = 0$, $s = 0$, *the sheet* S_η *of the focal surface degenerates, if* $r' = 0$, $s' = 0$ *the sheet* S_ξ *degenerates. If* $r = r' = s = s' = 0$ *both sheets of the surface become curves;* if $t^2 - 4kl$ also vanishes, these two focal curves coincide.

If the system of differential equations be reduced to its canonical form (68) for $t^2 - 4kl \neq 0$, if $r = 0$, μ must vanish, and the first equation becomes integrable by quadratures. Similarly the second for $r' = 0$.

Therefore one of the equations of our system becomes integrable by quadratures if, when written in the canonical form, either r and s or r' and s' vanish, i. e. if one of the sheets of the focal surface reduces to a curve. Both equations are integrable by quadratures if r, s and r', s' both vanish, i. e. if both sheets of the focal surface reduce to curves. This presupposes however a reduction to the canonical form, which requires the factoring of the quadratic covariant C (equation (67)), and the solution of equation (61), which is equivalent to the solution of the ordinary differential equation (60a).

Let us construct a canonical form for the case $t^2 - 4kl = 0$. We can assume for the first equation the same canonical form as before, so that our system will certainly have the form

$$by_2 + ey + fz = 0, \quad b \neq 0,$$
$$a'y_1 + b'y_2 + c'z_1 + d'z_2 + e'y + f'z = 0,$$

whence $\qquad k = 0, \quad l = bd', \quad m = 0, \quad n = -bc', \quad t = bc'.$

But in this case $t^2 - 4kl = 0$, so that c' must vanish. But then we can eliminate y_2 from the second equation by means of the first, which gives the canonical form sought

$$\left.\begin{aligned} y_2 &= \lambda y + \mu z, \\ a'y_1 + d'z_2 &= \nu y + \rho z. \end{aligned}\right\} \tag{73}$$

By differentiation we find

$$\mu d' y_{22} = \mu^2 a' y_1 + [d'(\mu_2 + \lambda\mu) - \rho\mu] y_2 + [(\lambda_2\mu - \mu_2\lambda)d' + \mu(\lambda\rho - \mu\nu)] y, \tag{74}$$

which proves that *the lines of a congruence, the sheets of whose focal surface coincide,*

are tangents to one of the two sets of asymptotic lines of that surface. For, any four solutions of an equation of the form

$$\theta_{22} = a\theta_1 + b\theta_2 + c\theta$$

define a surface upon which the curves $x_1 = \text{const.}$ are asymptotic lines.*

An exceptionally interesting example can be given for such systems $\Omega = 0$, $\Omega' = 0$, for which both sheets of the focal surface degenerate. For, consider the system, well known in the theory of functions,

$$\Omega = \frac{\partial y}{\partial x_2} + \frac{\partial z}{\partial x_1} = 0, \quad \Omega' = \frac{\partial y}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0, \tag{75}$$

for which we find

$$i = -1, \quad i' = -1, \quad k = -1, \quad l = -1, \quad m = n = t = 0, \quad t^2 - 4kl = -4$$

In order to make i and i' vanish, we consider the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

instead, where we put

$$\begin{aligned} \phi &= \sqrt{-1}, & \psi &= 1, \\ \chi &= -\sqrt{-1}, & \omega &= 1, \end{aligned}$$

which gives us

$$\frac{\partial y}{\partial x_1} + \sqrt{-1}\,\frac{\partial y}{\partial x_2} + \sqrt{-1}\,\frac{\partial z}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0,$$

$$\frac{\partial y}{\partial x_1} - \sqrt{-1}\,\frac{\partial y}{\partial x_2} - \sqrt{-1}\,\frac{\partial z}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0,$$

for which

$$k = -2\sqrt{-1}, \quad l = -2\sqrt{-1}, \quad m = -2, \quad n = -2, \quad t = 0, \quad t^2 - 4kl = 16,$$
$$i = i' = r = s = r' = s' = 0,$$

so that its focal surface degenerates into two distinct curves.

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* Darboux. Théorie des Surfaces, Tome I, p. 144.

To find these curves we make the substitution obtained by factoring the covariant C, i. e. we put

$$y = \sqrt{-1}\eta + \sqrt{-1}\zeta, \quad z = \eta - \zeta,$$

which gives

$$\sqrt{-1}\frac{\partial \eta}{\partial x_1} - \frac{\partial \eta}{\partial x_2} = 0, \quad \sqrt{-1}\frac{\partial \zeta}{\partial x_1} + \frac{\partial \zeta}{\partial x_2} = 0, \tag{76}$$

and therefore

$$\eta^{(k)} = f^{(k)}(x_1 + ix_2), \quad \zeta^{(k)} = g^{(k)}(x_1 - ix_2), \qquad (k = 1, 2, 3, 4), \tag{77}$$

where the $f^{(k)}$ and $g^{(k)}$ functions are arbitrary functions of the two independent arguments $x_1 + ix_2$ and $x_1 - ix_2$, so that equations (77) represent two arbitrary curves, whose points are put into an arbitrary correspondence with each other.

Therefore, the partial differential equations of the theory of functions represent any congruence whatever, which is made up of the lines intersecting two distinct curves. Any system of partial differential equations of form (1), *for which* $i = i' = r = r' = 0$, *can be transformed into the canonical form*

$$\frac{\partial y}{\partial x_2} + \frac{\partial z}{\partial x_1} = 0, \quad \frac{\partial y}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0.$$

We can make another interesting application. Let us suppose that $t^2 - 4kl \neq 0$, and that the sheet S_y of the focal surface is a surface of translation, which may be generated either by the curves $x_1 = \text{const.}$ or $x_2 = \text{const.}$ Moreover let us assume that our coordinates are cartesian. Then the first equation of (69) must reduce to $y_{12} = 0$. We must, therefore, have

$$\lambda = 0, \quad \mu_1 + \mu\rho = 0, \quad \lambda_1 + \mu\nu = 0,$$

for μ cannot vanish, since the surface S_y would otherwise degenerate into a curve, as shown by (68). Therefore, we find $\nu = 0$, i. e. the second sheet of the focal surface is a curve. We can therefore enunciate the following theorem:

Consider the congruences formed by the tangents of the curves $x_1 = $ *const. and* $x_2 = $ *const. of a surface of translation, the surface being capable of being generated*

by the translation of either a curve $x_1 = const.$ or $x_2 = const.$ All of the lines of such a congruence intersect a curve.

This theorem is essentially the converse of a theorem of Lie's.*

§9.—*Catalogue of types.*

We have seen that, if $l^2 - 4kl \neq 0$, system (1) can be reduced to the form

$$y_2 = \lambda y + \mu z, \quad z_1 = \nu y + \rho z.$$

But this can be simplified. For, if we put

$$y = \bar{y} \,.\, e^{\int \lambda dx_2}, \quad z = \bar{z} e^{\int \rho dx_1},$$

we find for $\bar{y}$ and $\bar{z}$ a system of the form

$$\bar{y}_2 = \bar{\mu}\bar{z}, \quad \bar{z}_1 = \bar{\nu}\bar{y}. \tag{78}$$

If $l^2 - 4kl = 0$, we have already found for the system the form

$$\left.\begin{aligned} y_2 &= \lambda y + \mu z, \\ a'y_1 + d'z_2 &= \nu y + \rho z. \end{aligned}\right\} \tag{73}$$

Let us assume $d' \neq 0$, and put

$$y = \bar{y} e^{\int \lambda dx_2}, \quad z = \bar{z} e^{\int -\frac{\rho}{d'} dx_2}.$$

Then (73) becomes

$$\bar{y}_2 = \bar{\mu}\bar{z}, \quad \bar{a}\bar{y}_1 + \bar{z}_2 = \bar{\nu}\bar{y}_1. \tag{79}$$

If $d' = 0$, (73) becomes

$$y_2 = \lambda y + \mu z, \quad a'y_1 = \nu y + \rho z.$$

* Lie-Scheffers, Geometrie der Berührungstransformationen, p. 383.

We may now assume $a' \neq 0$, for otherwise we could at once reduce our system of equations to a single equation of the first order. We may therefore put

$$y = \bar{y} e^{\int \frac{\nu}{a'} dx_1}, \quad z = \bar{z},$$

which gives

$$\bar{y}_2 = \bar{\lambda}\bar{y} + \bar{\mu}\bar{z}, \quad \bar{y}_1 = \bar{\rho}\bar{z}. \tag{80}$$

In this case the congruence degenerates. For we have

$$-\bar{\mu}\bar{y}_1 + \bar{\rho}\,\bar{y}_2 = \bar{\rho}\bar{\lambda}\bar{y}, \quad \bar{\rho}\bar{z} = \bar{y}_1.$$

The first equation shows that the focal surface is, in this case, a curve. The second shows that P_z is always on the tangent to this curve constructed at P_y. Instead of ∞^2 straight lines, we have therefore merely ∞^1, the tangents of the curve described by P_y. By writing $\bar{\rho}\bar{z} = z'$ and then dropping the strokes, we can write for this type more simply

$$y_1 = z, \quad y_2 = \lambda y + \mu z. \tag{81}$$

We have found, therefore, the following typical forms for systems of partial differential equations of form (1):

$$\left.\begin{array}{llll} \text{I.} & t^2 - 4kl \neq 0, & y_2 = \mu z, & z_1 = \nu y; \\ \text{II.} & t^2 - 4kl = 0, & y_2 = \mu z, & a y_1 + z_2 = \nu y; \\ \text{III.} & t^2 - 4kl = 0, & y_1 = z\ , & y_2 = \lambda y + \mu z. \end{array}\right\} \tag{82}$$

Any system of form (1), which cannot be reduced to a single equation, can be reduced to one of these forms. It now becomes a simple matter to decide whether two such systems are equivalent. The systems are reduced to their typical forms and these are compared. If the types are equivalent, the same thing is true of the original systems.

§10.—*Laplace transformations.*

A linear differential equation of the form

$$\frac{\partial^2 z}{\partial x\, \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0, \tag{83}$$

can be transformed into another of the same form by either of the transformations

$$z_1 = \frac{\partial z}{\partial y} + az, \quad z_{-1} = \frac{\partial z}{\partial x} + bz.^* \tag{84}$$

Thus from any equation of this form a single infinity of other equations is obtained by the transformation of Laplace. The original equation has two invariants

$$h = \frac{\partial a}{\partial x} + ab - c, \quad k = \frac{\partial b}{\partial y} + ab - c,$$

and if either of these is zero, the equation can be integrated by quadratures. Each of the equations, obtained from the given one by the method of Laplace, of course, also has two invariants, and if either of these vanishes, that equation and, therefore, the original one can be integrated by quadratures. A great many interesting investigations can be attached to the theory of these transformations.

M. Darboux has interpreted the transformation geometrically, in a beautiful manner, and it is on the basis of this geometrical interpretation that we shall introduce Laplace transformations of our system.

If $z^{(1)} \ldots . z^{(4)}$ are 4 solutions of (83) and are taken as the homogeneous coordinates of a point in space, then the locus of this point is a surface upon which the curves $x =$ const., $y =$ const., form a conjugate system. Consider the curves $x =$ const. and construct their tangents. They form a congruence one sheet of whose focal surface is the given surface. Clearly the tangents along a curve $x =$ const. form a developable surface of the congruence whose cuspidal edge is that curve $x =$ const. But we know that the lines of the congruence can be assembled into developable surfaces in another way. Consider any fixed curve $y =$ const. It intersects the curves $x =$ const. At each of the points of intersection, construct the tangent to the curve $x =$ const. at that point. These lines of course belong to the congruence, and form a developable surface whose cuspidal edge lies on the other sheet of the focal surface. Now it is this other sheet which is obtained by putting

$$z_1 = \frac{\partial z}{\partial y} + az.$$

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* Darboux, Théorie des surfaces. T. II, Chap. 1.

In the same way the second transformation arises from the congruence of lines tangent to the curves $y = \text{const.}$

Our transformations are precisely the same as these, geometrically. We assume that $t^2 - 4kl \neq 0$, so that we may write for our system

$$\frac{\partial y}{\partial x_2} = \mu z, \quad \frac{\partial z}{\partial x_1} = \nu y. \tag{85}$$

The surfaces S_y and S_z are the distinct sheets of the focal surface of our congruence C. The curves $x_1 = \text{const.}$ on S_y and the curves $x_2 = \text{const.}$ on S_z are the cuspidal edges of the two families of developable surfaces. We wish to deduce from the given congruence C another one C_1 which shall have the surface S_z in common with C as one sheet of its focal surface. But we shall assume that in this second congruence the curves $x_1 = \text{const.}$ on S_z instead of $x_2 = \text{const.}$ shall be the cuspidal edges of the one set of developable surfaces. Every line of the new congruence is then tangent to a curve $x_1 = \text{const.}$ on the surface S_z and, therefore joins P_z to a point on such a tangent, which we shall call P_ζ. We must therefore put

$$\eta = z, \quad \zeta = \alpha z + \frac{\partial z}{\partial x_2}. \tag{86}$$

But as P_z moves along a curve $x_2 = \text{const.}$, the line $P_z P_\zeta$ must describe a developable surface of the congruence C, which has its edge of regression on the, as yet unknown, second sheet of its focal surface. We wish to determine λ so that P_ζ shall be a point of this edge of regression. This is so if, and only if, during this motion the line $P_z P_\zeta$ always remains tangent to the curve described by P_ζ, i. e. if an equation of the form

$$\frac{\partial \zeta}{\partial x_1} = \beta z + \gamma \zeta \tag{87}$$

be fulfilled. If we substitute for ζ the value from (86), we find

$$z_{12} + \alpha z_1 + \alpha_1 z = \beta z + \gamma \alpha z + \gamma z_2,$$

or

$$z_{12} + \alpha z_1 - \gamma z_2 + (\alpha_1 - \beta - \gamma \alpha) z = 0.$$

But, on the other hand, we have, if we put $\lambda = \rho = 0$ in (69),

$$\nu z_{12} = \nu_2 z_1 + \mu \nu^2 z$$

or

$$z_{12} - \frac{\nu_2}{\nu} z_1 - \mu \nu z = 0.$$

The two equations must be identical, as they represent the same surface S, referred to the same conjugate system. Therefore,

$$\alpha = -\frac{\nu_2}{\nu}, \quad \gamma = 0, \quad \alpha_1 - \beta = -\mu\nu,$$

whence we see that the transformation (86) becomes

$$\eta = z, \quad \zeta = -\frac{\nu_2}{\nu} z + \frac{\partial z}{\partial x_2}. \tag{88}$$

Equations (87) and (86) give

$$\frac{\partial \zeta}{\partial x_1} = \left[\mu\nu - \frac{\partial^2 \log \nu}{\partial x_1 \partial x_2}\right]\eta, \quad \frac{\partial \eta}{\partial x_2} = \zeta + \frac{\partial \log \nu}{\partial x_2}\eta.$$

If we introduce $\frac{1}{\nu}\eta$ in place of η, we obtain the following result. The congruence C_1 is obtained by making the transformation

$$\eta = \frac{z}{\nu}, \quad \zeta = -\frac{\nu_2}{\nu} z + \frac{\partial z}{\partial x_2}, \tag{89}$$

and its equations are

$$\frac{\partial \eta}{\partial x_2} = \frac{1}{\nu}\zeta, \quad \frac{\partial \zeta}{\partial x_1} = \left(\mu\nu - \frac{\partial^2 \log \nu}{\partial x_1 \partial x_2}\right)\nu\eta. \tag{89a}$$

Similarly another congruence C_{-1} is obtained by putting

$$\eta = -\frac{\mu_1}{\mu} y + \frac{\partial y}{\partial x_1}, \quad \zeta = \frac{y}{\mu}, \tag{90}$$

viz.

$$\frac{\partial \eta}{\partial x_2} = \left(\mu\nu - \frac{\partial^2 \log \mu}{\partial x_1 \partial x_2}\right)\mu\zeta, \quad \frac{\partial \zeta}{\partial x_1} = \frac{1}{\mu}\eta. \tag{90a}$$

The repetition of the first transformation gives rise to a sequence of congruences C_1, C_2, C_3, ... and the second gives rise to a similar sequence C_{-1}, C_{-2}, C_{-3}, ... etc. If this sequence is finite in either direction, one of the sheets of the focal surface of the last congruence thus obtained degenerates into a curve. This takes place for C_1 for instance if

$$\mu\nu - \frac{\partial^2 \log \nu}{\partial x_1 \partial x_2} = 0.$$

It should be noted that for our system, the quantities r, s, r', s', become

$$r = 0, \quad s = -\mu, \quad r' = +\nu, \quad s' = 0,$$

so that clearly the coefficients of the various systems obtained by Laplace transformations are expressible in terms of r' and s, which take the place of the quantities h and k in the theory of the differential equation of Laplace.

The entire theory of the differential equation of Laplace can clearly be presented from this new point of view.

Moreover it should be noted that, although we have used the canonical form of our system of differential equations to deduce the form of the Laplace transformations, we might have used its unreduced form. Then it is clear that the various dependent variables introduced by the successive Laplace transformations will be connected covariantly with our system of partial differential equations.

PARIS, *September* 15th, 1903.

On Elements Connected each to each by One or the other of Two Reciprocal Relations.

By C. De Polignac.

CHAPTER I.

Art. 1. In what follows I represent by $\bar{\bar{E}}$ an aggregate of n elements.

$$e_1 e_2 \ldots . e_n$$

looked upon as mere ideal concepts.

Between any two elements there will exist by hypothesis a reciprocal relation R_1 or a reciprocal relation R_2 which, for pure convenience, I shall call respectively Variation and Permanence, without attaching any particular meaning to either of these terms. Except being reciprocal, these relations are undefined and unrestricted, so that if two elements are supposed to be connected to a third by the relation R_1, it does not follow that the same relation R_1 subsists between them as would be the case if, for instance, R_1 was a relation of equality or any other implying with respect to the individual element some intrinsic attribute of magnitude, form, etc.

Conventional Terms.

Any two elements e_i, e_j must, by the fundamental hypothesis, be connected either by a variation or by a permanence. If by a variation, for instance, we shall say that e_i and e_j have a variation together, or that e_j has a variation with e_i, and vice versa.

We shall also say that e_j is a variation-element of e_i, and vice versa.

It will sometimes be convenient to use the term variation instead of varia-

tion-element, thus, where no confusion of ideas can occur, we shall speak of the variations of e_i, meaning the variation-elements.

All the same expressions will apply to permanences, *mutatis mutandis.*

Cycles of Variations and Permanences.

Every pair of elements out of the aggregate $\bar{\bar{E}}$, having been arbitrarily given a variation, or a permanence, if several elements $ee' \dots e''$ have nothing but variations between them, we shall say that they form *a cycle of variations.* The term *cycle of permanences* will be similarly understood.

2.—*Annotation.*

We write the elements in columns with the only restriction that any two elements must not be put down in the same column when they are connected by *one* of the two relations, say to fix ideas, the relation called variation. The law of annotation is then simply:

Any two elements between which there is a variation, must not be written in the same column.

Regard being had to this rule, a number of schemes will be formed, such as

$$\begin{array}{llll} e_1 & e_2 & e_4 \dots & e_i, \\ e_3 & e_6 & \dots & e_j, \\ e_5 & & \dots & e_h, \\ \dots & \dots & \dots & \dots \end{array}$$

each scheme containing all the elements, and, in each column, the elements giving nothing but permanences.

Observe that, in conformity with the rule and whatever be the set of relations, we shall have the scheme:

$$e_1 \quad e_2 \quad e_3 \dots e_{n-1} \quad e_n,$$

in which each column is composed of a single element.

The law of annotation is only concerned with the composition of the columns in every scheme, not with the order in which the elements are written in each column, nor with the order in which the columns are written in each scheme. It must be, therefore, well understood that two schemes are distinct only when they differ by the composition of some column or other.

3.—*Table of Schemes.*

I shall, in what follows, use the word Table to designate an aggregate of annotated schemes. The full or general Table will mean the aggregate of all the schemes possible under the law of annotation. Their number necessarily depends upon the relations arbitrarily established between the elements, that is to say, upon the incidence of variation or permanence between each and every pair.

Ex. 1. By way of illustration, let us take seven elements, 1, 2, 3, 4, 5, 6, 7, connected as indicated in the subjoined synopsis, in which the variations belonging to every element are written on the same line as that element.

Elements.	Variations.				
1	2	3	6	7	
2	1	3	5	6	
3	1	2	4	5	
4	3	5	6	7	
5	2	3	4	6	
6	1	2	4	5	7
7	1	4	6		

The permanences belonging to each element are at once inferred from its variations, e. g. since the variations assigned to element 1 are with the elements 2, 3, 6, 7, it must have permanences with all the remaining elements, viz. 4, 5 (Art. 1), and so on.

From the adopted set of relations, the following cycles of variations are inferred:

Cycles of Variations.

1 2 3; 1 2 6; 1 6 7; 2 3 5; 2 5 6; 3 4 5; 4 5 6; 4 6 7;

all of three elements, and as, under the rule of annotation, no two elements giving a variation can enter the same column, it is clear that every scheme must have three columns at least.

In order to obtain all the schemes, we can begin by writing all the elements in a row,

1, 2, 3, 4, 5, 6, 7,

which gives the scheme having the maximum number of columns; then, by

referring to the synopsis of variations, we can reduce the number of columns successively.

Or, we can start with any cycle of variation, e. g.

$$1, \quad 2, \quad 3$$

and introduce the remaining elements one by one.

The application of either process will lead to the following general Table of twenty-seven schemes:

1.	$\begin{matrix}(1&2&3&5)\\(4&7&6&\;)\end{matrix}$	10.	$\begin{matrix}(1&2&3&5&6)\\(4&&&7&\;)\end{matrix}$	19.	$\begin{matrix}(1&2&3&4&5)\\(&&6&&7)\end{matrix}$
2.	$\begin{matrix}(1&2&3&5)\\(4&&6&7)\end{matrix}$	11.	$\begin{matrix}(1&2&3&6&7)\\(5&4&&&\;)\end{matrix}$	20.	$\begin{matrix}(1&2&3&5&6&7)\\(4&&&&&\;)\end{matrix}$
3.	$\begin{matrix}(1&2&3&7)\\(5&4&6&\;)\end{matrix}$	12.	$\begin{matrix}(1&2&3&4&6)\\(5&7&&&\;)\end{matrix}$	21.	$\begin{matrix}(1&2&3&4&6&7)\\(5&&&&&\;)\end{matrix}$
4.	$\begin{matrix}(1&2&3&6)\\(5&4&7&\;)\end{matrix}$	13.	$\begin{matrix}(1&2&3&4&7)\\(5&&6&&\;)\end{matrix}$	22.	$\begin{matrix}(1&2&3&5&6&7)\\(&4&&&&\;)\end{matrix}$
5.	$\begin{matrix}(1&2&3&4)\\(5&7&6&\;)\end{matrix}$	14.	$\begin{matrix}(1&2&3&4&6)\\(5&&7&&\;)\end{matrix}$	23.	$\begin{matrix}(1&2&3&4&5&6)\\(&7&&&&\;)\end{matrix}$
6.	$\begin{matrix}(1&2&3&5)\\(&4&6&7)\end{matrix}$	15.	$\begin{matrix}(1&2&3&5&7)\\(&4&6&&\;)\end{matrix}$	24.	$\begin{matrix}(1&2&3&4&5&7)\\(&&6&&&\;)\end{matrix}$
7.	$\begin{matrix}(1&2&3&5&6)\\(4&7&&&\;)\end{matrix}$	16.	$\begin{matrix}(1&2&3&5&6)\\(&4&7&&\;)\end{matrix}$	25.	$\begin{matrix}(1&2&3&4&5&6)\\(&&7&&&\;)\end{matrix}$
8.	$\begin{matrix}(1&2&3&5&6)\\(4&&7&&\;)\end{matrix}$	17.	$\begin{matrix}(1&2&3&5&6)\\(&4&&7&\;)\end{matrix}$	26.	$\begin{matrix}(1&2&3&4&5&6)\\(&&&&7&\;)\end{matrix}$
9.	$\begin{matrix}(1&2&3&5&7)\\(4&&6&&\;)\end{matrix}$	18.	$\begin{matrix}(1&2&3&4&5)\\(&7&6&&\;)\end{matrix}$	27.	$(1\ 2\ 3\ 4\ 5\ 6\ 7)$

4. Reverting to the general case, it is evident that the *minimum* number of columns met with in one or several schemes depends essentially upon the *connection* of the elements, that is to say, upon the particular way in which the variations and permanences are distributed. For the purpose of reference to this minimum, it will be convenient to adopt a letter for it. Accordingly, in what follows:

The minimum number of columns will be designated by k.

As an immediate consequence of the meaning of k, in connection with the law of annotation, the maximum cycle of variations cannot involve more than k elements. This consequence may also be stated thus: *Between $k+1$ elements taken at random, there exists at least one permanence.*

Observe that if the maximum cycle of variation contains k'-elements, it does not follow that $k=k'$. For, in the above example, the maximum cycle of variation has only three elements, yet $k=4$.

5. If we single out from the general Table all the schemes which have only k or $k+1$ columns, we shall obtain a reduced or partial table which we shall designate by T, and in which every scheme is annotated with the *minimum* number of columns, or the *minimum plus one.*

In the above example, Table T is made up of the schemes 1, 2, ... 19.

Abridged Mode of Writing.

In the sequel we shall represent an aggregate of schemes of an equal number of columns, say μ, by the abridged notation

$$|E_1\ E_2\ E_3 \ldots\ E_{\mu-1}\ E_\mu|,$$

where each E stands for a column or, what is the same thing, for some cycle of permanences in the widest sense of the term, including *pairs* or *single elements.* The symbols E_1, E_2, etc., must be understood to represent in turn the different cycles of permanences met with in the various schemes of μ columns, whose collective representation is condensed into the above abridged form.

According to this convention, Table T will be represented by the two abridged forms:

$$|E_1\quad E_2 \ldots E_k\ E_{k+1}|,$$
$$|\,.\ E_1\ E_2 \ldots E_k|,$$

the second of which gives the schemes of k columns considered as schemes of $k+1$ columns, with one blank column indicated by a dot.

It must be well borne in mind that E_1, E_2, etc., being mere symbols, do not represent the same aggregates of elements in both forms.

6. The representative form

$$|E_1\ E_2\ \ldots.\ E_k\ E_{k+1}|$$

admits of one *arbitrary* column, that is to say, in the aggregate of the schemes of $k+1$ columns, every cycle of permanences, including pairs and single elements, will, in some scheme, make up a whole column. This is easily seen as follows:

Let us designate by the notation $\dot{E}$ a cycle of permanences arbitrarily chosen, which might be a permanence-pair or a single element, and let us select any particular scheme of k columns, say

$$|.\ E_1\ E_2\ \ldots.\ E_{k-1}\ E_k|.$$

As this scheme, like every scheme, contains all the elements, we can pick out of the different columns, the elements of which $\dot{E}$ is composed, and place them together in the blank column, which gives the required scheme:

$$|\dot{E}\ E_1'\ E_2'\ \ldots.\ E_{k-1}'\ E_k'|.$$

Exceptionally this scheme might reduce to one of k columns. If, for instance, we had taken $\dot{E}=E_1$, our operation on the particular scheme of k columns originally selected would have been nothing else than the shifting of a whole column bodily, thus leaving the scheme unaltered. Nevertheless, we cannot assert *a priori* that a particular cycle of permanences will make up a whole column in some scheme of k columns, e. g. in the above example, where $k=4$, there is no scheme of four columns in which the element 3 makes up a whole column.

With schemes of $k+2$ columns, two columns would become arbitrary, and so on.

We can now condense the two representative forms of Table T into one, viz.

$$|\dot{E}\ E_1\ E_2\ \ldots.\ E_{k-1}\ E_k| \tag{1}$$

for this typical scheme has generally $k+1$ columns, and by making $\dot{E}=0$ we get the schemes of k columns in their abridged notation.

7.—*First Partition.*

Representing, as in Article 1, by $\bar{E}$, the aggregate of the given elements $e_1, e_2, \ldots. e_n$, we partition $\bar{E}$ into two arbitrary distinct sets of elements which

by analogy we shall designate by $\overline{A}$ and $\overline{B}$ respectively. We have then symbolically

$$\overline{E} = \overline{A} + \overline{B}.$$

In scheme (1) the symbol E becomes generally $\frac{A}{B}$; likewise the conventional symbol $\dot{E}$ becomes generally $\left(\frac{\dot{A}}{B}\right)$, i. e. a *chosen cycle of permanences containing elements of both sets.* We may, however, without ambiguity, write

$$\dot{E} = \frac{\dot{A}}{\dot{B}},$$

for it is obvious that once the particular cycle of permanences $\dot{A}$ has been chosen, $\dot{B}$ is limited by the law of annotation to such cycles of permanences out of the set $\overline{B}$ as form a cycle of permanences with $\dot{A}$ and *vice versa.* Otherwise told, the two cycles $\dot{A}$ and $\dot{B}$ are arbitrary, subject to the condition of being what may be called *congruent* with one another.

The general form of scheme (1) under the partition is

$$\begin{vmatrix} \dot{A} & A_1 \ldots\ldots A_{k-1} & A_k \\ \dot{B} & B_1 \ldots\ldots B_{k-1} & B_k \end{vmatrix}.$$

We can, in the arbitrary column, make separately or conjointly $\dot{A} = 0$, $\dot{B} = 0$. For $\dot{E} = \dot{A}$, $\dot{E} = \dot{B}$, $\dot{E} = 0$ are legitimate values of $\dot{E}$ in scheme (1).

If we make $\dot{B} = 0$, we obtain the following type of schemes:

$$\left| \begin{matrix} \dot{A} & A_1 & A_2 \ldots\ldots A_{k-1} & A_k \\ \cdot & B_1 & B_2 \ldots\ldots B_{k-1} & B_k \end{matrix} \right., \qquad (2)$$

comprising all the schemes of table T in which the elements of the set $\overline{B}$ occupy k columns at most.

The aggregate of such schemes forms a partial table which we shall call the *Table of the first Partition* and designated by T_1.

As all the schemes of T_1 are also schemes of T, we can say that table T_1 is contained in table T.

8. Table T_1 will also contain schemes of the type:

$$\left|\begin{array}{cccccc} A_1 & A_2 & A_3 \ldots\ldots & A_{k-1} & A_k & . \\ . & B_1 & B_2 \ldots\ldots & B_{k-2} & B_{k-1} & B_k \end{array}\right|, \tag{3}$$

in which both sets are written in k columns at most, two of which are different.

In order to justify this assertion, let us first write the scheme in the more convenient form

$$\begin{array}{lcccccc} \text{Columns} & 1 & 2 & 3 \ldots\ldots & k-1 & k & k+1 \\ \hline & A & A & A \ldots\ldots & A & A & . \\ & . & B & B \ldots\ldots & B & B & B \end{array}$$

in which it must be well understood that A, B are mere indeterminate symbols having a *different meaning in each column*, as representing in each an aggregate of different elements. All the symbols A, taken together, make up the set of elements designated by $\bar{A}$; similarly with B.

Now, if in (2), we make $\dot{A} = 0$, we get the general type of the schemes of k columns illustrating the partition, viz:

$$\left|\begin{array}{cccc} . & A_1 & A_2 \ldots\ldots & A_k \\ . & B_1 & B_2 \ldots\ldots & B_k \end{array}\right|.$$

We write it likewise:

$$\begin{array}{cccc} 1 & 2 & 3 \ldots\ldots & k+1 \\ \hline . & A & A \ldots\ldots & A \\ . & B & B \ldots\ldots & B \end{array}$$

and it can represent any particular scheme of k columns. If, now, we remove from any column, e. g. the second, all the elements which belong to the set $\bar{A}$ to place them in the first column—the blank column—we obtain

$$\begin{array}{cccccc} 1 & 2 & 3 \ldots\ldots & & k & k+1 \\ \hline A & . & A \ldots\ldots & & A & A \\ . & B & B \ldots\ldots & & B & B \end{array}$$

and this is a scheme of the type (3), for by permuting the second and the $k+1^{\text{th}}$ columns, which leaves the scheme unaltered (Art. 1), we obtain scheme (3) in the new abridged writing.

The two representative schemes of table T_1 of the first partition are then:

$$\begin{array}{cccc} 1 & 2 \ldots\ldots & k & k+1 \\ \hline \dot{A} & A \ldots\ldots & A & A \\ \cdot & B \ldots\ldots & B & B \end{array}, \tag{2}$$

$$\begin{array}{cccc} 1 & 2 \ldots\ldots & k & k+1 \\ \hline A & A \ldots\ldots & A & \cdot \\ \cdot & B \ldots\ldots & B & B \end{array}. \tag{3}$$

Scheme (2) alone admits of an arbitrary column confined to cycles of permanences chosen among the elements of the set $\overline{A}$.

9.—*Second Partition.*

We can proceed to a second partition by sub-partitioning the set $\overline{A}$ into two sets $\overline{A}_1$, $\overline{B}_1$. We have then symbolically

$$\overline{A} = \overline{A}_1 + \overline{B}_1, \quad \overline{E} = \overline{A}_1 + \overline{B}_1 + \overline{B}.$$

In carrying out this second partition in the two representative schemes of the first partition, the symbol B remains unchanged, while each symbol A changes to $\frac{A_1}{B_1}$. Therefore scheme (2) becomes

$$\begin{array}{cccc} 1 & 2 \ldots\ldots & k & k+1 \\ \hline \dot{A}_1 & A_1 \ldots\ldots & A_1 & A_1 \\ \dot{B}_1 & B_1 \ldots\ldots & B_1 & B_1 \\ \cdot & B \ldots\ldots & B & B \end{array}$$

the symbols A_1, B_1, B having as before different meanings in the different columns.

We can again make separately or simultaneously $\dot{A}_1 = 0$, $\dot{B}_1 = 0$ (comp. Art. 7). Making $\dot{B}_1 = 0$, we obtain the type of schemes

$$\begin{array}{cccc} 1 & 2 \ldots\ldots & k & k+1 \\ \hline \dot{A}_1 & A_1 \ldots\ldots & A_1 & A_1 \\ \cdot & B_1 \ldots\ldots & B_1 & B_1 \\ \cdot & B \ldots\ldots & B & B \end{array} \tag{4}$$

Likewise scheme (3) becomes

$$\begin{array}{cccc} 1 & 2 \;\ldots\; & k & k+1 \\ \hline \end{array}$$
$$\left|\begin{array}{cccc} A_1 & A_1 \;\ldots\; & A_1 & \cdot \\ B_1 & B_1 \;\ldots\; & B_1 & \cdot \\ \cdot & B \;\ldots\; & B & B \end{array}\right|. \tag{5}$$

Now, in the same way as we obtained a reduced table T_1 by singling out of table T,* all the schemes in which the elements of the set $\overline{B}$ occupied k columns at most, we can obtain another reduced table by singling out of T_1 after the sub-partition of $\overline{A}$ into $\overline{A}_1 + \overline{B}_1$ all the schemes in which the elements of the set B_1 occupy k columns at most.

We shall by analogy call this new reduced table the table of the second partition and designate it by T_2.

All the schemes of the types (4) or (5) belong to T_2. But I say that it will, in addition to these, include schemes of the three following types:

$$\begin{array}{cccc} 1 & 2 \;\ldots\; & k & k+1 \\ \hline \end{array}$$
$$\left|\begin{array}{cccc} A_1 & A_1 \;\ldots\; & A_1 & \cdot \\ \cdot & B_1 \;\ldots\; & B_1 & B_1 \\ \cdot & B \;\ldots\; & B & B \end{array}\right|, \tag{6}$$

$$\begin{array}{cccc} 1 & 2 \;\ldots\; & k & k+1 \\ \hline \end{array}$$
$$\left|\begin{array}{cccc} A_1 & A_1 \;\ldots\; & A_1 & \cdot \\ \cdot & B_1 \;\ldots\; & B_1 & B_1 \\ B & B \;\ldots\; & B & \cdot \end{array}\right|, \tag{7}$$

$$\begin{array}{cccc} 1 & 2 \;\ldots\; & k & k+1 \\ \hline \end{array}$$
$$\left|\begin{array}{cccc} A_1 & A_1 \;\ldots\; & A_1 & A_1 \\ B_1 & B_1 \;\ldots\; & B_1 & \cdot \\ \cdot & B \;\ldots\; & B & B \end{array}\right| \tag{8}$$

---

* Let us recall that table T is, by definition, made up of the aggregate of schemes of no more than $k+1$ columns.

The proof is as follows:

If in type (4) we make $\dot{A}_1 = 0$, we have a form illustrating the second partition in any scheme of k columns, viz.

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & \ldots\ldots & k & k+1 \end{array} \\ \hline \left|\begin{array}{ccccc} . & A_1 & \ldots\ldots & A_1 & A_1 \\ . & B_1 & \ldots\ldots & B_1 & B_1 \\ . & B & \ldots\ldots & B & B \end{array}\right| . \end{array}$$

Removing the elements belonging to the set $\overline{A}_1$ from the last column to the blank column, we obtain the identical scheme type (6).

To justify the existence of the scheme (7) in table T_2, we first observe that it can be written

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & \ldots\ldots & k & k+1 \end{array} \\ \hline \left|\begin{array}{ccccc} A_1 & A_1 & \ldots\ldots & A_1 & . \\ B & B & \ldots\ldots & B & . \\ . & B_1 & \ldots\ldots & B_1 & B_1 \end{array}\right| , \end{array}$$

for the permutation of the two rows is tantamount to a permutation of elements in each column and leaves the scheme unaltered (Art. 2).

This done, it is only necessary to imagine the first partition to have been

$$\overline{\overline{E}} = \overline{\overline{\mathfrak{A}}} + \overline{B}_1$$

and the sub-partition

$$\mathfrak{A} = \overline{A}_1 + \overline{B},$$

which gives the same total partition of elements as before, viz.

$$\overline{\overline{E}} = \overline{A}_1 + \overline{B} + \overline{B}_1.$$

Type (7) is then the new form of type (5).

The existence of type (8) follows from the definition of table T_2; for it is obvious that the schemes in which both the set $\overline{B}$ and the set $\overline{B}_1$ take up individually k columns at most, will generally contain the remaining elements, i. e. those of the set $\overline{A}_1$ in all the columns. The same type can also be inferred from the types (5) or (7) by removing into the column devoid of the elements of the set $\overline{A}_1$ one or more elements of that set, which may happen to form a cycle of permanences with the elements of the set $\overline{B}$ or the set $\overline{B}_1$ contained in that column.

The table T_2 of the second partition is then characterized by the five types, (4), (5), (6), (7), (8).

They might also be represented by graphs, e. g.

$$\begin{array}{c|l|} \overline{A}_1 & \rule{3cm}{0.4pt}\ \cdot \\ \overline{B}_1 & \cdot\ \rule{3cm}{0.4pt} \\ \overline{B} & \cdot\ \rule{3cm}{0.4pt} \end{array}\,, \tag{6}$$

$$\begin{array}{c|l|} \overline{A}_1 & \rule{3cm}{0.4pt}\ \cdot \\ \overline{B}_1 & \cdot\ \rule{3cm}{0.4pt} \\ \overline{B} & \rule{3cm}{0.4pt}\ \cdot \end{array}\,, \tag{7}$$

etc.

10. There is no difficulty in generalizing these results. We can form a new reduced table T_3 by singling out of table T_2 the schemes in which the elements of another set $\overline{B}_2$ occupy at most k columns. T_3 will be the table of the third partition of the given elements expressed by the symbolical equation

$$\overline{E} = \overline{A}_2 + \overline{B}_2 + \overline{B}_1 + \overline{B},$$

and this third partition can be conceived to have been obtained from the second partition by sub-partitioning $\overline{A}_1$ into $\overline{A}_2 + \overline{B}_2$.

The typical schemes in any new partition will be inferred from those of the preceding one by breaking up the various symbols into two as may be observed by comparing the first partition with the second. In any representative scheme we may have any number of symbols excluded from one column and the remaining ones from the other, or also one symbol subject to no exclusion, as in type (8) of the second partition, e. g.

$$\begin{array}{cccc} 1 & 2 & \ldots\ldots\ k & k+1 \\ \hline \multicolumn{4}{|c|}{} \\[-2ex] A_m & A_m & \ldots\ldots\ A_m & A_m \\ \cdot & B_m & \ldots\ldots\ B_m & B_m \\ B_{m-1} & B_{m-1} & \ldots\ldots\ B_{m-1} & \cdot \\ \vdots & \vdots & \ldots\ldots\ \vdots & \vdots \\ \cdot & B_1 & \ldots\ldots\ B_1 & B_1 \\ B & B & \ldots\ldots\ B & \cdot \end{array}$$

where the symbol A_m can be struck out from the first or from the last column.

If, in any representative scheme, we condense into one symbol all the symbols for which the dot occurs in the same column, we obtain, as should be, a representative scheme belonging to the first partition, for this operation is tantamount to condensing several sets into one, so as to reduce the total number of sets to two.

Remark.—There is only one type of schemes in any partition where one column may be considered arbitrary, viz.

$$\begin{array}{c|cccccc|}
1 & 2 & & \ldots & k & k+1 \\
\hline
\dot{A}_m & A_m & & \ldots & A_m & A_m \\
\cdot & B_m & B_m & \ldots & B_m & B_m \\
\cdot & B_{m-1} & B_{m-1} & \ldots & B_{m-1} & B_{m-1} \\
 & \vdots & \vdots & \ldots & \vdots & \vdots \\
\cdot & B_1 & B_1 & \ldots & B_1 & B_1 \\
\cdot & B & B & \ldots & B & B
\end{array}.$$

This circumstance will play no part in the sequel, and while it seemed difficult to omit every mention of it, on account of its obviousness, yet no particular stress need be laid upon it.

11. The results arrived at so far can be summed up in the following proposition:

Proposition I.—If elements in any number be arbitrarily connected each to each by one or the other of two reciprocal relations, and if k be the minimum of columns in which they can be annotated (subject to the law given, Art. 2), then if the aggregate of these elements be partitioned into any number of sets arbitrarily composed, it will always be possible to annotate schemes of no more than $k+1$ columns in which the elements belonging to any number of these sets shall be excluded from one column, and the elements belonging to the remainder of the sets shall be excluded from another column.

Otherwise,

In building up schemes of no more than $k+1$ columns by introducing the elements one by one in any order, it will always be possible to obtain schemes in which every element as, and when introduced, shall be excluded from one or from the other of two columns of previously determined ranks.

Remark.—The possibility of such an exclusion is strictly confined to two columns. In the second partition, for instance, we have not been led to any scheme of the type

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots\ldots & k & k+1 \\ \hline \cdot & A_1 & A_1 & \ldots\ldots & A_1 & A_1 \\ B_1 & \cdot & B_1 & \ldots\ldots & B_1 & B_1 \\ B & B & \cdot & B \ldots\ldots & B & B \end{array}$$

in which the three sets appear, simultaneously excluded from three different columns. Nor is the existence of such a scheme generally possible so long as the connection between the elements remains arbitrary.

Calling *exclusion feature* the general possibility of excluding sets of elements from one or more columns, Proposition I asserts that—

When the connection of the elements remains arbitrary, the exclusion-feature is, generally speaking, restricted to two columns with respect to every element in the aggregate of the schemes of no more than $k+1$ columns, where k is the minimum of columns which any scheme can have.

12. *Ex.* 2.—Let us take thirteen elements connected as follows:

Elements.	Variations.							
1	2	3	4	6	7	8	13	
2	1	3	5	6	8	9		
3	1	2	4	5	7	9		
4	1	3	5	6	8	11		
5	2	3	4	6	7	8	13	
6	1	2	4	5	7	10	11	
7	1	3	5	6	8	9	12	13
8	1	2	4	5	7	9	10	
9	2	3	7	8	10	12		
10	6	8	9	11	13			
11	4	6	10	12				
12	7	9	11					
13	1	5	7	10				

In order to determine k (minimum of columns, Art. 4), we can start with the cycle of variations

$$1 \quad 2 \quad 3$$

and introduce the elements one by one in their natural numerical order, putting them down in the smallest number of columns consistent with the synopsis of variations. It is only necessary to take into consideration the variation of each successive element with the preceding elements already annotated.

e. g. From the synopsis, element 4 has variations with elements 1, 3 already annotated; we can then write:

$$\begin{array}{ccc} 1 & 2 & 3 \\ & 4 & \end{array}$$

Element 5 has variations with 2, 3, 4, and we can write

$$\begin{array}{ccc} 1 & 2 & 3 \\ 5 & 4 & \end{array}$$

Proceeding in this way, we shall obtain a *unique* scheme of three columns, viz.

$$\left|\begin{array}{ccc} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 9 & 7 & 8 \\ 11 & 10 & 12 \\ & & 13 \end{array}\right| .$$

We have then $k = 3$.

In order to illustrate the typical schemes of the second partition, let us throw the thirteen elements into three sets:

$$\begin{aligned} \bar{A}_1 &= 1 \quad 2 \quad 3 \quad 11 \quad 13, \\ \bar{B}_1 &= 5 \quad 7 \quad 8 \quad 10, \\ \bar{B} &= 4 \quad 6 \quad 9 \quad 12, \end{aligned}$$

which we can conceive to be derived from a first partition

$$\begin{aligned} \bar{A} &= 1 \quad 2 \quad 3 \quad 5 \quad 7 \quad 8 \quad 10 \quad 11 \quad 13, \\ \bar{B} &= 4 \quad 6 \quad 9 \quad 12, \end{aligned}$$

followed by the sub-partition of $\bar{A}$ into $\bar{A}_1 + \bar{B}_1$, where $\bar{A}_1$ and $\bar{B}_1$ are the sets above given.

If in the scheme of three columns we remove from the first column the elements 5, 9 which belong to $\overline{B}_1$ and $\overline{B}$ respectively, we obtain the scheme

$$\begin{array}{c|cccc|}
\overline{A}_1 & 1 & 2 & 3 & \cdot \\
 & 11 & & 13 & \\[1em]
\overline{B}_1 & \cdot & 7 & 8 & 5 \\
 & & 10 & & \\[1em]
\overline{B} & \cdot & 4 & 12 & 9 \\
 & & & 6 &
\end{array}$$

illustrating the typical scheme (6) of the second partition.

If we only remove element 5 into the fourth column, we get

$$\begin{array}{c|cccc|}
\overline{A}_1 & 1 & 2 & 3 & \cdot \\
 & 11 & & 13 & \\[1em]
\overline{B}_1 & \cdot & 7 & 8 & 5 \\
 & & 10 & & \\[1em]
\overline{B} & 9 & 4 & 6 & \cdot \\
 & & & 12 &
\end{array}$$

illustrating type (7).

If in this last scheme we remove 11 to the fourth column, we obtain a scheme of the type (8), viz.

$$\begin{array}{c|cccc|}
\overline{A}_1 & 1 & 2 & 3 & 11 \\
 & & & 13 & \\[1em]
\overline{B}_1 & \cdot & 7 & 8 & 5 \\
 & & 10 & & \\[1em]
\overline{B} & 9 & 4 & 6 & \cdot \\
 & & & 12 &
\end{array}$$

Remark.—Although always leading to a specimen of types (5)(6)(7), this simple process may not be sufficient to obtain all the schemes of a given type;

e. g. the scheme

$$\begin{array}{c|cccc|}
\overline{A}_1 & 1 & 2 & 3 & . \\
 & & 11 & 13 & \\
\\
\overline{B}_1 & 5 & 7 & 8 & . \\
 & 10 & & & \\
\\
\overline{B} & . & & 6 & 4 \\
 & & & 12 & 9
\end{array}$$

which illustrates type (5), cannot be evolved out of the scheme of three columns by the mere removal of a few elements from one and the same column, but it is easily obtained by introducing the elements successively.

The restriction of the *exclusion-feature* asserted in Proposition I and pointed out in Art. 11, Remark, can also be readily illustrated.

Let it be required to compose a scheme of no more than four columns in which the above sets, $\overline{A}_1$, $\overline{B}_1$, $\overline{B}$, shall be excluded from the first, the second and the third columns respectively.

We can start with the cycle of variation 1 2 3 and as these three elements are to be excluded from the first column as belonging to the set $\overline{A}_1$, we can write them in the six different ways:

$$| \; . \;\; 1 \;\; 2 \;\; 3 \; |; \quad | \; . \;\; 1 \;\; 3 \;\; 2 \; |; \quad | \; . \;\; 2 \;\; 1 \;\; 3 \; |;$$
$$| \; . \;\; 2 \;\; 3 \;\; 1 \; |; \quad | \; . \;\; 3 \;\; 1 \;\; 2 \; |; \quad | \; . \;\; 3 \;\; 2 \;\; 1 \; |;$$

which, in composing schemes, must be examined one after the other.

We begin with $| \; . \;\; 1 \;\; 2 \;\; 3 \; |$ and introduce the elements in their natural order.

Element 4 has the variations 1, 3 (Synopsis), and being excluded from the third column as belonging to the set $\overline{B}$, can only be put down in the first column. We have then so far

$$\begin{array}{c|cccc|l}
\overline{A}_1 & . & 1 & 2 & 3 & , \\
\overline{B}_1 & & . & & & , \\
\overline{B} & 4 & & . & & .
\end{array}$$

Element 5 has the variations 2, 3, 4, and being excluded from the second

column as belonging to the set $\overline{B}_1$, can be placed nowhere in this scheme, which consequently drops out.

The next initial annotation is

$$| \cdot \quad 1 \quad 3 \quad 2 |.$$

Here element 4 has two places, giving the two schemes,

$$\begin{array}{c} \overline{A}_1 \\ \overline{B}_1 \\ \overline{B} \end{array} \left| \begin{array}{cccc} \cdot & 1 & 3 & 2 \\ & \cdot & & \\ 4 & & \cdot & \end{array} \right|, \qquad \begin{array}{c} \overline{A}_1 \\ \overline{B}_1 \\ \overline{B} \end{array} \left| \begin{array}{cccc} \cdot & 1 & 3 & 2 \\ & \cdot & & \\ & & \cdot & 4 \end{array} \right|.$$

Again the first one does not admit of element 5. In the second, 5 has only one place, and we get

$$\begin{array}{c} \overline{A}_1 \\ \overline{B}_1 \\ \overline{B} \end{array} \left| \begin{array}{cccc} \cdot & 1 & 3 & 2 \\ 5 & \cdot & & \\ & & \cdot & 4 \end{array} \right|,$$

but 6 having the variations 1, 2, 4, 5, and being excluded from the third column as belonging to the set $\overline{B}$, can be placed nowhere, and the scheme is lost.

We must then turn to the third initial annotation of the elements 1, 2, 3 and by trying them all in succession, it will be found that in several schemes we can successfully annotate eight elements, but no more, e. g. we have the scheme

$$\begin{array}{c} \overline{A}_1 \\ \overline{B}_1 \\ \overline{B} \end{array} \left| \begin{array}{cccc} \cdot & 3 & 1 & 2 \\ 8 & \cdot & 5 & 7 \\ & 6 & \cdot & 4 \end{array} \right|$$

into which element 9, having the variations 2, 3, 7, 8, and being excluded from the third column as belonging to the set $\overline{B}$, cannot be annotated.

13. To sum up: Whatever be the distribution of variations and permanences among the given elements, we can, by means of the synopsis recording their connection, build up schemes, element by element, of k and $k+1$ columns (k, minimum number of columns, Art. 4), and if, at any stage of the process, we throw the elements already annotated into two distinct arbitrary sets $\overline{A}$ and $\overline{B}$, we know from the results obtained in this chapter and embodied in Proposi-

tion I, that among the schemes so far obtained there will be one or more of the type

$$\begin{array}{ccccc} 1 & 2 & 3 \ldots\ldots & k & k+1 \\ \hline \end{array}$$
$$\left|\begin{array}{ccccc} A & A & A \ldots\ldots & A & . \\ . & B & B \ldots\ldots & B & B \end{array}\right|,$$

and again, it follows therefrom that in introducing the next element into the schemes of this type, it will be possible, in one or more, to exclude said element, at will, from the column which contains no element of the set $\bar{A}$ or from the column which contains no element of the set $\bar{B}$; for this additional element might have been *a priori* assigned to the set $\bar{A}$ or to the set $\bar{B}$.

CHAPTER II.

14. Hitherto the connection between the elements was purely arbitrary. We shall now examine what restrictions must be introduced in order to obtain a full extension of the exclusion-feature described in Proposition I.

k being as before the minimum number of columns met with in any scheme, (Art. 4), the desired extension consists in the possibility of composing schemes of $k+1$ columns in which $k+1$ arbitrary sets of elements shall be excluded respectively from $k+1$ different columns.

Adhering to former notations, the symbol of such schemes would be (Chapter I, Art. 8),

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots\ldots k-1 & k & k+1 \\ \hline \end{array}$$
$$\left|\begin{array}{cccccc} . & A_{k-1} & A_{k-1} & \ldots\ldots A_{k-1} & A_{k-1} & A_{k-1} \\ B_{k-1} & . & B_{k-1} & \ldots\ldots B_{k-1} & B_{k-1} & B_{k-1} \\ B_{k-2} & B_{k-2} & . & \ldots\ldots B_{k-2} & B_{k-2} & B_{k-2} \\ \vdots & & & & & \vdots \\ B_2 & B_2 & B_2 & \ldots\ldots . & B_2 & B_2 \\ B_1 & B_1 & B_1 & \ldots\ldots B_1 & . & B_1 \\ B & B & B & \ldots\ldots B & B & . \end{array}\right|, \qquad \text{(S)}$$

or, more generally,

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots\ldots & k-1 & k & k+1 \\ \hline \multicolumn{1}{|c}{A_k} & A_k & A_k & \ldots\ldots & A_k & A_k & \multicolumn{1}{c|}{A_k} \\ \multicolumn{1}{|c}{\cdot} & B_k & B_k & \ldots\ldots & B_k & B_k & \multicolumn{1}{c|}{B_k} \\ \multicolumn{1}{|c}{B_{k-1}} & \cdot & B_{k-1} & \ldots\ldots & B_{k-1} & B_{k-1} & \multicolumn{1}{c|}{B_{k-1}} \\ \multicolumn{1}{|c}{\vdots} & & & & & & \multicolumn{1}{c|}{\vdots} \\ \multicolumn{1}{|c}{B_2} & B_2 & B_2 & \ldots\ldots & \cdot & B_2 & \multicolumn{1}{c|}{B_2} \\ \multicolumn{1}{|c}{B_1} & B_1 & B_1 & \ldots\ldots & B_1 & \cdot & \multicolumn{1}{c|}{B_1} \\ \multicolumn{1}{|c}{B} & B & B & \ldots\ldots & B & B & \multicolumn{1}{c|}{\cdot} \end{array}, \qquad (S')$$

where one of the sets, viz. $\overline{A}_k$ is subject to no restriction as to annotation.

That such schemes are obtainable I will presently show, but the exposition will be facilitated by the introduction of a few new conventional terms.

15.—*Monogeneous and Polygeneous Elements.*

When schemes are built up element by element, each additional element finds a certain number of variations and of permanences among the elements already annotated.

If the new element has only one variation among the elements already annotated, it will be called *monogeneous;* if it has several, *polygeneous,* and in the latter case, *duogeneous, trigeneous,* and, generally, *m-geneous,* according to the number of such variations, or also *polygeneous* of the order m.

16. We can now assert the following fact:

If an additional element be $k-1$-geneous at most, it will be possible to exclude it from any column whatsoever in each of the schemes of $k+1$ columns previously annotated.

The proof is immediate.

By hypothesis, the new element is $k-1$-geneous at most. By definition, it finds in each previous scheme at most $k-1$ variations. The elements giving these variations can then take up at most $k-1$ columns in each scheme. Hence, schemes of $k+1$ columns being allowed, the $k-1$-geneous element can

be entered into two columns at least in each scheme. In other words, there are in each scheme two columns at least in which the $k-1$-geneous element finds nothing but permanences.

This told, one of two things must happen.

Either in any scheme the particular column from which it is intended to exclude the new element, contains one of its variations, or it does not.

If it does, the new element is by the law of annotation excluded from that column, and the extra prohibitive condition is superfluous.

If it does not, the new element might be put down in such column, but may be also kept off from it and entered elsewhere, as there is at least one other column in which the new element has no variation. Q. E. D.

The same will be true of another $k-1$-geneous element, and, consequently, of any number of $k-1$-geneous consecutive elements.

As an immediate consequence, if the given elements can be assigned such an order as will make each at most $k-1$-geneous with respect to the preceding ones, then in the annotation of the schemes the possibility of exclusion will be unrestricted.

17. There is no difficulty in obtaining such an aggregate of elements. The easiest way is to start with a cycle of variations of k elements

$$e_1 \quad e_2 \quad e_3 \cdots\cdots e_{k-1} \quad e_k,$$

when, so far, the minimum of columns is assuredly k, and next to add any number of elements

$$e_{k+1}, \quad e_{k+2} \cdots\cdots e_{n-1}, \quad e_n$$

at most $k-1$-geneous, which is done by giving e_{k+1} no more than $k-1$ variations among the preceding elements $e_1, e_2 \ldots\ldots e_k$; in the same way giving e_{k+2} no more than $k-1$ variations among the preceding elements, $e_1, e_2, \ldots\ldots e_k, e_{k+1}$, and so on. The value of k as designating the minimum of columns will remain unaltered throughout, and in the schemes of $k+1$ columns the exclusion-feature, as before understood, will be unrestricted.

18. *Ex.* 3.—As an example, let us resume the thirteen elements of example 2 (Art. 12), in which $k=3$. They are not *duogeneous* ($k-1$-geneous) in their

natural order, but if we strike out the elements 5, 6, 8, 9, 13 the remaining eight elements, 1, 2, 3, 4, 7, 10, 11, 12 are at most *duogeneous* in the order in which they are given, as the synopsis in Ex. 2 shows.

We now add twelve more duogeneous elements, and write down the new synopsis:

SYNOPSIS.

ELEMENTS.	VARIATIONS.	
Duogeneous at most.	With preceding elements.	With subsequent elements.
1	none	2 3 4 7
2	1	3
3	1 2	4 7 14
4	1 3	11
7	1 3	12
10	none	11 16 20
11	4 10	12 15 16
12	7 11	14 18
14	3 12	15 19 22
15	11 14	17 20
16	10 11	17 18 21
17	15 16	19
18	12 16	21
19	17 14	23
20	10 15	22
21	18 16	24
22	14 20	23 25
23	19 22	24
24	21 23	25
25	22 24	none.

We still have $k = 3$, for the synopsis shows that the following scheme of three columns is relevant under the law of annotation:

$$\left|\begin{array}{ccc} 1 & 2 & 3 \\ 10 & 4 & 11 \\ 12 & 7 & 17 \\ 15 & 14 & 18 \\ 19 & 16 & 20 \\ 21 & 23 & 24 \\ 22 & 25 & \end{array}\right| .$$

Take now
$$\begin{array}{lccccc} \overline{A}_2 = & 1 & 2 & 3 & 11 & 25 \\ \overline{B}_2 = & 7 & 10 & 16 & 19 & 23 \\ \overline{B}_1 = & 14 & 15 & 18 & 21 & 24 \\ \overline{B} = & 4 & 12 & 17 & 20 & 22 \end{array}$$

and let it be proposed to work out a scheme of the type:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline \end{array}$$
$$\left|\begin{array}{cccc} . & A_2 & A_2 & A_2 \\ B_2 & . & B_2 & B_2 \\ B_1 & B_1 & . & B_1 \\ B & B & B & . \end{array}\right| .$$

Starting with the embryo-scheme $| \; . \; 1 \; 2 \; 3 \; |$, and introducing the elements one by one in the numerical order of magnitude, we obtain, among others, the scheme:

$$\overline{A}_2 \left|\begin{array}{cccc} . & 1 & 2 & 3 \\ & 25 & & 11 \end{array}\right| ,$$

$$\overline{B}_2 \left|\begin{array}{cccc} 16 & & 7 & 19 \\ 23 & . & 10 & \end{array}\right| ,$$

$$\overline{B}_1 \left|\begin{array}{cccc} 14 & 15 & . & 18 \\ & 21 & & 24 \end{array}\right| ,$$

$$\overline{B} \left|\begin{array}{cccc} 4 & 12 & 17 & . \\ 20 & & 22 & \end{array}\right| .$$

19. The unrestricted exclusion-feature which, as proven, pertains to an aggregate of elements, at most $k-1$-geneous *inter se* is expressed in signs by the two symbolical schemes (S), (S′) of Art. 14. We can in (S′) put $A_k = 0$ in any column, and by condensing into one the two symbols which will then be missing from the same column we fall back on the type (S). In both the symbols can be further sub-divided into any number of symbols.

We can now sum up the results obtained in this chapter in the following proposition:

Proposition II.—If the given elements can be assigned such an order as will make them at most $k-1$-geneous, *inter se*, the exclusion-feature in the aggregate of schemes of $k+1$ columns is unrestricted.

CHAPTER III.

20.—I propose in this chapter to draw some inferences from Proposition I (Ch. 1, Art. 11).

Let $\bar{E}$, as before, be an aggregate of elements arbitrarily connected. Making first partition, we have

$$\bar{E} = \bar{A} + \bar{B},$$

and, as proven in Chapter I, k being the minimum of columns, we shall find among the schemes of no more than $k+1$ columns schemes of the type

$$\begin{array}{ccccc} 1 & 2 \ldots. & k & k+1 \\ \hline \end{array}$$
$$\left| \begin{array}{cccc} A & A \ldots. & A & A \\ . & B \ldots. & B & B \end{array} \right|, \qquad (2)$$

which is symbolical scheme (2) of the first partition (Ch. I, Art. 7).

N. B.—We have left out the dot over the first symbol A as immaterial, and will do so in the future. (See Remark, Art. 10.)

Let us now introduce a new element α, giving it permanences with all the elements of the set $\bar{A}$ and variations with all the elements of the set $\bar{B}$. We can enter α into the column which contains no element of the set $\bar{B}$ in all the schemes

of the type (2) whose aggregate forms what in Ch. I has been called the table of the first partition. We could enter α into no other scheme without augmenting the number of columns, and as it is understood all along that we reject schemes with more than $k+1$ columns, all the schemes not of the type (2) drop out, and after the introduction of α, the general representative scheme of the table under consideration is

$$\begin{array}{cccc} 1 & 2 \ldots\ldots & k & k+1 \\ \hline \multicolumn{4}{|l|}{\alpha} \\ \multicolumn{1}{|c}{A} & A \ldots\ldots & A & \multicolumn{1}{c|}{A} \\ \multicolumn{1}{|c}{.} & B \ldots\ldots & B & \multicolumn{1}{c|}{B} \end{array} . \qquad (2)$$

21.—*First Sub-partition.*

The aggregate of elements being now $\overline{A} + \alpha + \overline{B}$, we introduce a second new element, α_1, giving it any number of variations but with the express condition that they be all taken out of $\overline{A} + \alpha$. The elements thus selected compose a set which we shall call $\overline{B}_1$, the remaining elements, another set $\overline{A}_1$, and we have then

$$\overline{A} + \alpha = \overline{A}_1 + \overline{B}_1.$$

By construction, α_1 has nothing but permanences in both the sets, $\overline{A}_1$ and $\overline{B}$ As for α, it may be assigned either to $\overline{A}_1$ or to $\overline{B}_1$; we leave it undecided for the present.

I say now that, *without augmenting the number of columns*, it will be possible to annotate α_1 in some of the schemes in which α has already been annotated.

This, as will be seen, results at once from Proposition I; among others we shall obtain at least one scheme of the form

$$\begin{array}{cccc} 1 & 2 \ldots\ldots & k & k+1 \\ \hline \multicolumn{1}{|c}{} & & & \multicolumn{1}{c|}{\alpha_1} \\ \multicolumn{1}{|c}{A_1} & A_1 \ldots\ldots & A_1 & \multicolumn{1}{c|}{A_1} \\ \multicolumn{4}{|l|}{\alpha} \\ \multicolumn{1}{|c}{B_1} & B_1 \ldots\ldots & B_1 & \multicolumn{1}{c|}{.} \\ \multicolumn{1}{|c}{.} & B \ldots\ldots & B & \multicolumn{1}{c|}{B} \end{array} \qquad (8)$$

for, if we erase both α and α_1, we get the scheme type (8) of the second partition, the concrete existence of which in one or more specimens has been proven. (Ch. I, Art. 9.) But in every such scheme α and α_1 are here relevantly annotated, under the rule laid down, viz. all the variations of α make up the set $\overline{B}$, those of α_1, the set B_1.

The elements of the set B do not participate in this second partition, which can be looked upon as a sub partition of the set $\overline{A}$, increased by α, into $\overline{A}_1$ and $\overline{B}_1$. We have already used the term sub-partition in Ch. I, and it will be convenient to continue to do so, the second partition being the first sub-partition, and so on.

In scheme type (2) of the original partition, we have written α over the symbol A, and in scheme type (8) of the first partition, α_1 over the symbol A_1. This, in order to avoid confusion, and to indicate that α in scheme (2) and α_1 in scheme (8), are *free* elements, i. e. yet unassigned to any particular set.

From the place which α occupies in scheme type (8), it can be conceived to belong either to $\overline{A}_1$ or to $\overline{B}_1$. In the latter case only there is a variation between α and α_1.

If α has been assigned to the set $\overline{A}_1$, then α and α_1 give a permanence, and we shall have such schemes as

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & \cdots & k & k+1 \end{array} \\
\hline
\left|\begin{array}{ccccc}
\alpha_1 & & & & \\
A_1 & A_1 & \cdots & A_1 & A_1 \\
\alpha & & & & \\
\cdot & B_1 & \cdots & B_1 & B_1 \\
\cdot & B & \cdots & B & B
\end{array}\right|
\end{array},
$$

which, by erasing α and α_1, reduce to the schemes type (4) of the second partition (Ch. I, Art. 9), the concrete existence of which has been proven. Moreover, α and α_1 are relevantly annotated in respect of their connection both with one another and with the original elements.

22.—*Second Sub-partition.*

The aggregate of elements is now

$$\overline{A}_1 + \alpha_1 + \overline{B}_1 + \overline{B},$$

α being included in $\overline{A}_1$ or in $\overline{B}_1$. We introduce a third new element, α_2, giving it any number of variations, but all taken out of the set $\overline{A}_1 + \alpha_1$.

These variations form a set $\overline{B}_2$, the remainder a set $\overline{A}_2$. We have then

$$\overline{A}_1 + \alpha_1 = \overline{A}_2 + \overline{B}_2,$$

and, by construction, α_2 has permanences with all the elements of the three sets, $\overline{A}_2$, $\overline{B}_1$, $\overline{B}$; α and α_1 may conjointly or separately belong either to $\overline{A}_2$ or to $\overline{B}_2$.

Here again it will be possible, without augmenting the number of columns, to put down α_2 in some of the schemes which already contain α and α_1.

To show it clearly, observe that the connection between α, α_1, α_2 depends on the sets to which α and α_1 have respectively been assigned in the first, and in the second, sub-partitions. Due regard being had to the rule governing the introduction of the new elements, α, α_1, α_2, we find in all the six following cases:

1°. α, α_1, α_2 form a cycle of permanences. Then in the first sub-partition, $A + \alpha = \overline{A}_1 + \overline{B}_1$, α must have been assigned to $\overline{A}_1$, and in the second, $A_1 + \alpha = \overline{A}_2 + \overline{B}_2$, both α and α_1 must have been assigned to $\overline{A}_2$.

2°.	α	α_1.	Permanence.
	α_1	α_2.	Id.
	α_2	α.	Variation.

Then in the first sub-partition, α has been assigned to $\overline{A}_1$. In the second, α_1 has been assigned to $\overline{A}_2$, α to $\overline{B}_2$.

3°.	α	α_1.	Permanence;	α	assigned to $\overline{A}_1$,	1st sub-partition.
	α_1	α_2.	Variation;	α_1	" " $\overline{B}_2$,	2d sub-partition.
	α_2	α.	Permanence;	α	" " $\overline{A}_2$,	" "
4°.	α	α_1.	Variation;	α	assigned to $\overline{B}_1$,	1st sub-partition.
	α_1	α_2.	Permanence;	α_1	" " $\overline{A}_2$,	2d sub-partition.
	α_2	α	Id.			
5°.	α	α_1.	Variation;	α	assigned to $\overline{B}_1$,	1st sub-partition.
	α_1	α_2.	Id.	α_1	" " $\overline{B}_2$	2d sub-partition.
	α_2	α.	Permanence.			
6°.	α	α_1.	Permanence;	α	assigned to $\overline{A}_1$,	1st sub-partition
	α_1	α_2.	Variation;	α_1	" " $\overline{B}_2$,	2d sub-partition.
	α_2	α.	Id.	α	" " $\overline{B}_2$,	

Several otherwise possible cases must here be ruled out, e. g. α, α_1, α_2 cannot form a cycle of variations, for a variation between α and α_1 implies that α has been assigned to $\overline{B}_1$, but, by the standing rule, α_2 has nothing but permanences in the sets $\overline{B}$, $\overline{B}_1$.

A relevant annotation of α_2 in the six cases can be illustrated; among others, in the six corresponding schemes below written:

$$\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & 2 & \cdots\cdots & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_2 & & & \\
A_2 & A_2 & \cdots\cdots & A_2 \\
\alpha_1 & & & \\
\alpha & & & \\
\cdot & B_2 & \cdots\cdots & B_2 \\
\cdot & B_1 & \cdots\cdots & B_1 \\
\cdot & B & \cdots\cdots & B
\end{array}\ 1^\circ,
\qquad
\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & 2 & \cdots\cdots & \multicolumn{1}{c}{k+1} \\
\hline
 & & & \alpha_2 \\
A_2 & A_2 & \cdots\cdots & A_2 \\
 & & & \alpha_1 \\
B_2 & B_2 & \cdots\cdots & \cdot \\
\alpha & & & \\
B_1 & B_1 & \cdots\cdots & \cdot \\
\cdot & B & \cdots\cdots & B
\end{array}\ 2^\circ,$$

$$\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & 2 & \cdots\cdots & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_2 & & & \\
A_2 & A_2 & \cdots\cdots & A_2 \\
\alpha & & & \\
\cdot & B_2 & \cdots\cdots & B_2 \\
 & & & \alpha_1 \\
B_1 & B_1 & \cdots\cdots & \cdot \\
\cdot & B & \cdots\cdots & B
\end{array}\ 3^\circ,
\qquad
\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & 2 & \cdots\cdots & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_2 & & & \\
A_2 & A_2 & \cdots\cdots & A_2 \\
 & & & \alpha_1 \\
\cdot & B_2 & \cdots\cdots & B_2 \\
B_1 & B_1 & \cdots\cdots & \cdot \\
\alpha & & & \\
\cdot & B & \cdots\cdots & B
\end{array}\ 4^\circ,$$

$$\begin{array}{|cccc|}
\multicolumn{1}{c}{1} & 2 & \cdots\cdots & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_2 & & & \\
A_2 & A_2 & \cdots\cdots & A_2 \\
\cdot & B_2 & \cdots\cdots & B_2 \\
 & & & \alpha_1 \\
B_1 & B_1 & \cdots\cdots & \cdot \\
\alpha & & & \\
\cdot & B & \cdots\cdots & B
\end{array}\ 5^\circ,
\qquad
\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \cdots\cdots & & \multicolumn{1}{c}{k+1} \\
\hline
 & & & & \alpha_2 \\
A_2 & A_2 & \cdots\cdots & A_2 & A_2 \\
B_2 & B_2 & \cdots\cdots & B_2 & \cdot \\
\alpha & & & & \\
\alpha_1 & & & & \\
\cdot & B_1 & \cdots\cdots & B_1 & B_1 \\
\cdot & B & \cdots\cdots & B & B
\end{array}\ 6^\circ.$$

The concrete existence of these symbolical schemes can be inferred from the schemes of the first sub-partition (Art. 21) by cancelling α_2 and condensing the two symbols A_2, B_2 into one A_1, when we fall back on the types (4) and (8) of the first sub-partition. Or directly leaving α, α_1, α_2 out of notice, the schemes actually exist in virtue of Proposition I. Moreover, α, α_1 and α_2 are relevantly annotated in respect of their connection, both with one another and with the original elements.

Observe that some cases can be illustrated by more than one representative scheme, e. g. in case 1° the six schemes are relevant; in case 2°, the schemes (2) and (6), etc.

23.— *Generalization.*

The possibility verified up to three additional elements must now be extended to any number of them.

We first recall the main features of the question (Comp., Arts. 20, 21, 22).

A first partition

$$\overline{E} = \overline{A} + \overline{B}$$

being made, the first additional element α is introduced. It is given permanences with all the elements of the set $\overline{A}$ and variations with all the elements of the set $\overline{B}$.

The sub-partition of rank i is

$$\overline{A}_{i-1} + \alpha_{i-1} = \overline{A}_i + \overline{B}_i.$$

It is made for the purpose of introducing the next additional element α_i, which is given permanences with all the elements of the set $\overline{A}_i$ and variations with all the elements of the set $\overline{B}_i$.

The elements of the set $\overline{B}_i$ do not participate in any further sub-partition, they can receive no further variations.

α_{i-1} can be assigned, at will, to $\overline{A}_i$ or to $\overline{B}_i$. In the latter case alone, it is connected with α_i by a variation.

From the rule governing the introduction of the new elements, as here restated, the following immediate inferences can be drawn :

Inferences.

i. The set $\bar{A}_i$ can include any element α_h, where $h < i$. In that case α_h must have been assigned successively to $\bar{A}_{h+1}$, $\bar{A}_{h+2}$ .... $\bar{A}_{i-1}$, $\bar{A}_i$; for if in any intermediate sub-partition α_h had been assigned to $\bar{B}_j$, where j is one of the numbers $h+1$, $h+2$, etc., it would not, by the rule, have participated in any further sub-partition, and could not, therefore, belong to $\bar{A}_i$.

Any other additional element α_j, where $j < i$, can also belong to $\bar{A}_i$, and generally, any number of additional elements or α-elements, as we will designate them, for brevity.

ii. All the α-elements belonging to $\bar{A}_i$ form a cycle of permanences. For if within $\bar{A}_i$, α_h and α_j gave a variation, where $h < j < i$, then α_h would belong to $\bar{B}_j$, and could not belong to $\bar{A}_i$.

This cycle of permanences admits also of α_i, as, by construction, every variation of α_i is included in $\bar{B}_i$.

iii. Any element α_i can have variations with any number of α-elements of ranks inferior to its own. For by ii $\bar{A}_{i-1}$ can include any number of previous α-elements, and these can, in the next sub-partition $\bar{A}_{i-1} + \alpha_{i-1} = \bar{A}_i + \bar{B}_i$, be all assigned to $\bar{B}_i$. They form a cycle of permanences by ii.

iv. Any α-element can be connected by a variation with only *one* α-element of a rank *superior* to its own.

For if α_i and $\alpha_{i+\mu}$ give a variation, α_i must have been assigned successively to $\bar{A}_{i+1}$, $\bar{A}_{i+2}$ .... $\bar{A}_{i+\mu-1}$, and, finally, to $\bar{B}_{i+\mu}$, when it can receive no more variations.

v. As a consequence, the first additional element α can have only one variation among the α-elements.

vi. The elements of the sets $\bar{B}$, $\bar{B}_1$, etc., must, in all generality, be assumed to actually occupy k columns in each scheme, from which it follows that the additional elements α, α_1, etc., can, generally speaking, be annotated in only *two* columns.

Assuming now the possibility of relevantly annotating m additional elements, α, α_1, ... α_{m-1} in two columns, whatever be the sets to which they may have been successively and are finally assigned, we shall establish the same possibility with respect to α_m, i. e. with $m+1$ additional elements.

To fix ideas, we take $m = 6$, but the reasoning will be general.

We have then to show that $7 = m + 1$ elements $\alpha, \alpha_1, \ldots. \alpha_6$ can be annotated in two columns under any distribution among the sets of the successive sub-partitions, e. g. the following:

$\alpha_2 \quad \alpha_4 \quad \alpha_5$ assigned to $\overline{A}_6$. Hence $(\alpha_2 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6)$ form a cycle of permanences from ii.

α_3 " " $\overline{B}_4$; $\alpha_4 \quad \alpha_3$. Variation. } Standing rule.
α_4 " " $\overline{B}_2$; $\alpha_2 \quad \alpha_1$. Id. }

We leave α *purposely* unassigned for the present.

The conventional partition-equations are:

Original partition, $\overline{E} = \overline{A} + \overline{B}$.

1st sub-partition,	$\overline{A} + \alpha = \overline{A}_1 + \overline{B}_1$,	α unassigned thus far.		
2d "	$\overline{A}_1 + \alpha_1 = \overline{A}_2 + \overline{B}_2$,	α_1	assigned to	$\overline{B}_2$.
3d "	$\overline{A}_2 + \alpha_2 = \overline{A}_3 + \overline{B}_3$,	α_2	"	$\overline{A}_3$.
4th "	$\overline{A}_3 + \alpha_3 = \overline{A}_4 + \overline{B}_4$,	$\begin{cases}\alpha_3 \\ \alpha_2\end{cases}$	" "	$\overline{B}_4$. $\overline{A}_4$.
5th "	$\overline{A}_4 + \alpha_4 = \overline{A}_5 + \overline{B}_5$,	$\begin{cases}\alpha_2 \\ \alpha_4\end{cases}$	" "	$\Big\}\overline{A}_5$.
6th "	$\overline{A}_5 + \alpha_5 = \overline{A}_6 + \overline{B}_6$	$\begin{cases}\alpha_2 \\ \alpha_4 \\ \alpha_5\end{cases}$	"	$\Big\}\overline{A}_6$.

But so long as α is unassigned, we have here virtually a distribution of $6 = m$ additional elements $\alpha_1, \alpha_2, \ldots. \alpha_6$ to an original aggregate $\overline{A} = \overline{E} - \overline{B}$. For cancelling α, the above equation of the first sub-partition becomes

$$\overline{A} = \overline{A}_1 + \overline{B}_1,$$

which represents an original partition of the aggregate $\overline{A}$. Likewise, the above equation of the second sub-partition, viz.

$$\overline{A}_1 + \alpha_1 = \overline{A}_2 + \overline{B}_2$$

becomes that of the first sub-partition to an original aggregate $\overline{A}$, and so on, the only difference being that the additional elements have here suffixes equal to their ranks, which is immaterial.

Hence, by hypothesis, there are one or more schemes corresponding to the above distribution of these $6 = m$ elements $\alpha_1, \alpha_2, \ldots\ldots \alpha_6$ in which they are relevantly annotated in two columns. Let us take any one of them, e. g.

$$\left|\begin{array}{llll}
\alpha_6 & & & \\
A_6 & A_6 \ldots\ldots & & A_6 \\
\alpha_5 & & & \\
\alpha_4 & & & \\
\alpha_2 & & & \\
\cdot & B_6 \ldots\ldots & & B_6 \\
\cdot & B_5 \ldots\ldots & & B_5 \\
\cdot & B_4 \ldots\ldots & & B_4 \\
 & & & \alpha_3 \\
B_3 & B_3 \ldots\ldots & B_3 & \cdot \\
\cdot & B_2 \ldots\ldots & & B_2 \\
 & \cdot & & \alpha_1 \\
B_1 & B_1 \ldots\ldots & B_1 & \cdot
\end{array}\right| . \qquad \text{(S)}$$

If we prove that we can reinstate the set $\overline{B}$ and the element α under any hypothesis as to its connection, we shall have established the existence of a proper scheme to the original aggregate $\overline{E}$ and $7 = m + 1$ additional elements, with an arbitrary distribution of these elements among the sets of the successive sub-partitions.

The proof is immediate.

By vi, α can have at most one variation among the α-elements, and this will happen if it has been finally assigned to some set $\overline{B}_i$, giving it a variation with α_i. Then, in the last written scheme, α can be written in the column first or last in which α_i is not, and the exclusion-column of the set $\overline{B}_i$ is thereby determined.

If, for instance, we take $i=3$, we have the scheme

$$\begin{array}{|lcc|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2 \;\ldots\; k} & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_6 & & \\
A_6 & A_6 \ldots A_6 & A_6 \\
\alpha_5 & & \\
\alpha_4 & & \\
\alpha_2 & & \\
\cdot & B_6 \ldots B_6 & B_6 \\
\cdot & B_5 \ldots B_5 & B_5 \\
\cdot & B_4 \ldots B_4 & B_4 \\
 & & \alpha_3 \\
B_3 & B_3 \ldots B_3 & \cdot \\
\alpha & & \\
\cdot & B_2 \ldots B_2 & B_2 \\
 & & \alpha_1 \\
B_1 & B_1 \ldots B_1 & \cdot \\
\cdot & B \ldots B & B
\end{array} \qquad (S_1)$$

If α has been finally assigned to the set $\overline{A}_m$, here $\overline{A}_6$, it has *no variation* among the α-elements and can be entered *ad lib.* into the first or the last column, giving the two schemes:

$$\begin{array}{|lcc|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{\ldots} & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_6 & & \\
A_6 & A_6 \ldots A_6 & A_6 \\
\alpha_5 & & \\
\alpha_4 & & \\
\alpha_2 & & \\
\alpha & & \\
\cdot & B_6 \ldots B_6 & B_6 \\
\cdot & B_5 \ldots B_5 & B_5 \\
\cdot & B_4 \ldots B_4 & \cdot \\
 & & \alpha_3 \\
B_3 & B_3 \ldots B_3 & \cdot \\
\cdot & B_2 \ldots B_2 & B_2 \\
 & & \alpha_1 \\
B_1 & B_1 \ldots B_1 & \cdot \\
\cdot & B \ldots B & B
\end{array} \qquad (S_2)$$

1	2	....	k	$k+1$
α^6				
A_6	A_6	....	A_6	A_6
α_5		....		α
α_4				
α_2				
.	B_6	....	B_6	B_6
.	B_5	....	B_5	B_5
.	B_4	....	B_4	B_4
				α_3
B_3	B_3	....	B_3	.
.	B_2	....	B_2	B_2
				α_1
B_1	B_1	....	B_1	.
B	B	....	B	.

(S_3)

The actual existence of these schemes can be vindicated *a posteriori* as in the second sub-partition (Art. 22). If we cancel the α-elements, the three schemes reduce to two and each has at least one actual specimen by Proposition I. On the other hand, the additional elements are correctly annotated in respect of their connection both with one another and with the original elements.

24.—*Definitions of the Terms "Outer Group," "Inner Group."*

We shall generally call "*outer group*" the aggregate of the elements which, in the process of the successive introduction of elements, remain susceptible of receiving new variations.

We shall call "*inner group*" the aggregate of the elements debarred from that possibility, whether arbitrarily or in consequence of some standing rule or hypothesis.

Here, under the rule laid down for the introduction of additional elements, after each new sub-partition, viz.

$$\overline{A}_{i-1} + \alpha_{i-1} = \overline{A}_i + \overline{B}_i,$$

made for the purpose of adding one or more element α_i, the outer group is

$$\overline{A}_i + \alpha_i,$$

the inner group, $\overline{B}_i + \overline{B}_{i-1} \ldots . \overline{B}_1 + \overline{B}.$

25. The introduction of these two new terms will facilitate the statement of the following facts:

I.—All the α-elements belonging to the outer group can be written in one and the same column.

This may be inferred from schemes (S) and (S_2) (Art. 23). If, before reinstating α into S, all the α-elements of the outer group could be written in the first column as in (S), α also can, as shown in (S_2).

A direct proof is also easily obtained.

Admit the assertion with m α-elements, $\alpha, \alpha_1, \ldots . \alpha_{m-1}$, and let α_p, α_q, etc., be the α-elements which have been finally assigned to $\overline{A}_{m-1}$; the α-elements of the outer group are then $\alpha_{m-1}, \alpha_p, \alpha_q$, etc., and we have, by hypothesis, the scheme

$$\left|\begin{array}{lll} \alpha_{m-1} & & \\ A_{m-1} & A_{m-1} \ldots . & A_{m-1} \\ \alpha_p & & \\ \alpha_q & & \\ \cdot & B_{m-1} \ldots . & B_{m-1} \\ \cdot & B_p \quad \ldots . & B_p \\ \cdot & B_q \quad \ldots . & B_q \\ \text{etc.} & & \end{array}\right| .$$

In the next sub-partition

$$\overline{A}_{m-1} + \alpha_{m-1} = \overline{A}_m + \overline{B}_m,$$

we can assign α_{m-1} either to $\overline{A}_m$ and the outer group or to $\overline{B}_m$ and the inner group, keeping in both cases $\alpha_p\, \alpha_q$, etc., in the outer group.

In the first case we have

$$
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & \ldots & k+1 \end{array} \\
\left|\begin{array}{llll}
\alpha_m & & & \\
A_m & A_m & \ldots\ldots & A_m \\
\alpha_{m-1} & & & \\
\cdot & B_m & \ldots\ldots & B_m \\
\alpha_p & & & \\
\alpha_q & & & \\
\text{etc.} & & & \\
\cdot & B_{m-1} & \ldots\ldots & B_{m-1} \\
\cdot & B_p & \ldots\ldots & B_p \\
\cdot & B_q & \ldots\ldots & B_q \\
\text{etc.} & & &
\end{array}\right|
\end{array} .
$$

In the second case,

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & \ldots\ldots & & k+1 \end{array} \\
\left|\begin{array}{lllll}
\alpha_m & & & & \\
A_m & A_m & \ldots\ldots & A_m & A_m \\
\alpha_p & & & & \\
\alpha_q & & & & \\
\text{etc.} & & & & \\
\cdot & B_m & \ldots\ldots & B_m & B_m \\
 & & & & \alpha_{m-1} \\
B_{m-1} & B_{m-1} & \ldots\ldots & B_{m-1} & \cdot \\
\cdot & B_p & \ldots\ldots & B_p & B_p \\
\cdot & B_q & \ldots\ldots & B_q & B_q \\
\text{etc.} & & & &
\end{array}\right|
\end{array} ,
$$

in both cases the assertion is justified with one more α-element.

II.—The elements of the outer group can be written in k columns.

For, in either of the two last written schemes, we can put $A_m = 0$ in the last column by Proposition I independently from the connection of the α-elements.

III.—The elements of the outer group continue to exhibit the exclusion-feature as defined Proposition I.

Before proceeding to the proof, let us observe that, after the introduction of the α-elements, the minimum of columns has became $k+1$; therefore, Proposition I does not hold in the aggregate of schemes under consideration; one more column would now be necessary. Neithor does it hold with respect to the α-elements *inter se*, for with respect to these we have $k=2$, and they are only allowed two columns. But, as seen in II, the elements of the outer group comprising an arbitrary set $\overline{A}_m$, made up of original elements and of any number of additional elements, α_p, α_q, etc., are in some schemes actually written in k-columns. Therefore, Proposition I would apply to the outer group if annotated separately in $k+1$ columns, and the condition involved in III is that we shall always find among the schemes resulting from the addition, under the rule laid down, of any number of elements α, α_1, etc., all the schemes which are necessary to illustrate the restricted exclusion-feature with regard to the elements of the outer group in all possible cases. Below a direct and general proof.

26.—*Demonstration of Assertion* III.

From the first wording of Proposition I (Ch. I, Art. 11), the exclusion-feature will be established with respect to the *elements of the outer group* if, throwing them into two sets, $\overline{G}$, $\overline{G}'$, arbitrarily composed, we prove that in the aggregate of the schemes obtained after the addition of any number m of the new elements, $\alpha, \alpha_1, \ldots. \alpha_{m-1}$, there exist schemes in which the elements composing $\overline{G}$, are written in k columns only, and the elements composing $\overline{G}'$, also in k columns, one of which is different; otherwise expressed schemes in which the elements of $\overline{G}$ and of $\overline{G}'$ are *missing* from two different columns, which can be taken as the first and the $k+1^{\text{th}}$.

The elements of the outer group $\overline{A}_m + \alpha_m$ are of two kinds: firstly, elements pertaining to the original aggregate $\overline{E}$, and, secondly, additional elements, viz. α_m and α_p, α_q, ... α_r, which have been finally assigned to $\overline{A}_m$.

Similarly, the two arbitrary sets, $\overline{G}$, $\overline{G}'$, will contain original and additional elements. One of the two sets $\overline{G}$ or $\overline{G}'$ must include α_m. Let us write

$$\overline{G} = \overline{C} + \alpha_p + \alpha_q \ldots. \alpha_r + \alpha_m;$$
$$G' = \overline{C}' + \alpha_{p'} + \alpha_{q'} + \ldots. \alpha_{r'},$$

$\overline{C}$ and $\overline{C}'$ being exclusively composed of original elements all contained in $\overline{A}_m$. The suffixes $p, q, \ldots . r$; $p', q', \ldots . r'$, are arbitrary except, of course, that every one of them is $< m$. One of them may be $= 0$; the corresponding element α_0 being in our notation identical with α.

To justify assertion III, we must find at least one scheme in which the elements composing $\overline{G}$ will be missing from say, the last column, and those composing $\overline{G}'$ from the first.

If such a scheme exists, the elements of the outer group must appear in it as shown in the following *embryo-scheme:*

$$
\begin{array}{|ccccc|}
\multicolumn{1}{c}{1} & 2 & \ldots . & k & \multicolumn{1}{c}{k+1} \\
\hline
\alpha_m & & & & \\
C & C & \ldots . & C & . \\
. & C' & \ldots . & C' & C' \\
\alpha_p & & \ldots . & & \alpha_{p'} \\
\alpha_q & & \ldots . & & \alpha_{q'} \\
\alpha_r & & \ldots . & & \alpha_{r'} \\
. & B_p & \ldots . & & B_p \\
. & B_q & \ldots . & & B_q \\
. & B_r & \ldots . & . & B_r \\
B_{p'} & & \ldots & B_{p'} & . \\
B_{q'} & & \ldots . & B_{q'} & . \\
B_{r'} & & \ldots . & B_{r'} & .
\end{array},
$$

and we have only to prove that the elements of the inner group can be relevantly inserted into it.

Let α_h be the additional element of the inner group which has the *highest suffix.* Being in the inner group, it belongs to some B-set (Art. 24), and has a variation with one, and only one, α-element of a rank superior to its own. (Art. 23, Inference iv.) This α-element is then in the outer group. Moreover, α_h has nothing but permanences with—

1°. The α-elements of the outer group whose ranks are inferior to its own.

For if α_h had a variation, e. g. with α_r, where $r < h$, α_r would belong to $\overline{B}_h$, a set of the inner group, contrary to our hypothesis.

2°. The elements involved in the symbols already written, viz. C, C', B_p, B_q, etc.

For, by the standing rule, the variations of α_h make up the set $\overline{B}_h$ and the corresponding symbol, B_h is not yet written in our scheme.

In conclusion, α_h has one, and only one, variation in the scheme as written so far, and can be entered into the column first or last, in which its variation-element is not.

If it be α_p for instance, we can write α_h in the last column with the symbol B_p, and at the same time the symbol B_h in all the columns *except* the last. We have then added one more α-element and one more line of symbols to our embryo-scheme, which now reads:

1	2	....	k	$k+1$
α_m				
C	C	....	C	.
.	C'	....	C'	C'
α_p		....		$\alpha_{p'}$
α_q		....		$\alpha_{q'}$
.	B_p	....		B_p
etc.				α_h
$B_{p'}$	$B_{p'}$	....	$B_{p'}$	.
etc.				
B_h	B_h	....	B_h	.

α_h being disposed of, let α_i be the additional element of the inner group whose suffix is now greatest. α_i, like α_h, belongs to some B-set, and has a variation with an α-element whose suffix is $> i$. It may be α_h or an α-element of the outer group. In any event α_i, just as before, can have only one variation among the elements already annotated. Hence, it can be placed in the column first or last in which its variation-element does not occur.

If, to fix ideas, we take α_h to be the variation-element of α_i, we must enter α_i into the first column with the symbol B_h, and exclude the symbol B_i from the first column. This done, we have again added one more α-element and one more

line of symbols to our scheme, which now reads:

1	2		k	$k+1$
α_m				
C	C	....	C	.
.	C'	....	C'	C'
α_p		....		$\alpha_{p'}$
α_q		....		$\alpha_{q'}$
α_r		....		$\alpha_{r'}$
.	B_p	....	B_p	B_p
				α_h
.	B_q	....	B_q	B_q
.	B_r	....	B_r	B_r
$B_{p'}$	$B_{p'}$	....	$B_{p'}$	.
$B_{q'}$	$B_{q'}$	....	$B_{q'}$	.
$B_{r'}$	$B_{r'}$	....	$B_{r'}$	.
B_h	B_h	....	B_h	.
α_i				
.	B_i	....	B_i	B_i

Proceeding in this way and reinstating the α-elements of the inner group in the order of decreasing ranks, each of them will always find one, and only one, variation among the elements already annotated, and can then be written in the column first or last, which does not contain the α-element, with which it has a variation.

When all the α-elements and the corresponding B-symbols have been annotated, we have a scheme which is *concrete* with respect to the additional elements, as each letter α represents only one element, and *symbolical* with respect to the original elements. The actual existence of such a scheme is justifiable *a posteriori* by the reasoning often made use of before; the symbolical part of the scheme has, from Proposition I, one or more concrete specimens, no matter how the sets $\overline{B}, \overline{B}_1, \ldots . \overline{B}_{m-1}, \overline{C}, \overline{C}'$ are composed, and, on the other hand, the additional elements $\alpha, \alpha_1, \ldots . \alpha_m$ are relevantly annotated both *inter se* and in relation to the original elements, as each additional element α_i is written in the column from which the symbol B_i, which represents the aggregate $\overline{B}_i$ of its variations, can be and has been excluded.

Assertion III is thus fully proven.

27. The process of sub-partitioning the outer group can go on as long as there are elements left to operate upon. At some stage of the addition of α-elements, the outer group might happen to be made up of these altogether. In any case at every stage the outer group is composed for elements which in one or more schemes occupy only k columns. Consequently we could introduce an element α_m giving it variations with *all the elements of the outer group.* This is tantamount to making $\overline{A}_m = 0$ in the equation of the m^{th} sub-partition

$$\overline{A}_{m-1} + \alpha_{m-1} = \overline{A}_m + \overline{B}_m .$$

We have then

$$\overline{A}_{m-1} + \alpha_{m-1} = \overline{B}_m ,$$

and the outer group reduces to the single element α_m. The corresponding symbolical scheme has the form

$$\begin{array}{|cccc|}
1 & 2 & \ldots . k & k+1 \\
\hline
\alpha_m & & & \\
\cdot & B_m & \ldots . B_m & B_m \\
 & & & \alpha_{m-1} \\
 & B_{m-1} & \ldots . B_{m-1} & \cdot \\
\text{etc.} & & &
\end{array}$$

28. We can now enunciate the following theorem:

THEOREM.—If elements, arbitrarily connected, have been annotated (according to the rule given Chapter I, Art. 2) in k and $k+1$ columns, k being the minimum, it will be possible, without augmenting the number $k+1$ of columns to add any number of elements, each with any number of variations, provided that at each addition, all the variation-elements of the new element pass on to the inner group. (Definition, Art. 24.)

At each addition many schemes may drop out, viz. those in which none of the $k+1$ columns admit of the introduction of the new element, but there will always remain one or more schemes into which it can be entered.

29. In illustrating this theorem, it will be more interesting to show its bearing on maps in connection with what has been called the geographical problem of the four colors.*

* A. B. Kempe, B. A. On the geographical problem of the four colors. American Journal of Mathematics, Vol. II.

The problem, as well known, is to determine how many colors are necessary to paint a map so that two adjacent districts are not painted the same color. By adjacent districts are understood those having a common line-boundary, not merely touching in a point, and the map is supposed to be drawn on a singly connected surface, say a sheet of paper.

Now the rule prescribing a change of color takes no account of any attributes, collective or individual, of the districts, whether in number, shape or space-magnitude. As regards the object aimed at, the districts in any map are mere elements connected each to each by one or the other of two reciprocal relations, which may be called:

Variation in the case of adjacent districts,
Permanence in the case of non-adjacent ones.

Here the second relation is simply the direct negative of the first. If we represent the districts by numbers or letters, the notion of coloring them with a certain number of colors under the rule stated, is the same as writing them in a certain number of columns subject to the condition that no two adjacent districts shall have their representative signs in the same column. Each column will then stand for a color.

From this point of view the coloring of a map is only a particular case of the annotation of elements treated of in this chapter and in the preceding ones, particular in that the connection of the elements is no longer arbitrary but subject to restrictions dependent on the configuration of maps, to which, however, such of the results so far obtained as are free from any hypothesis respecting the connection of the elements will be applicable.

The theorem of this chapter for instance, admits of an immediate adaptation.

A map has external districts which are the border districts, giving the map its *contour.* It has also generally internal districts surrounded by others, and completely shut in. To add a new district we can join two points arbitrarily taken on the contour by a line of any shape drawn outside the map and not meeting itself again. Such a line encloses a space which gives a new external district. The two points selected on the contour are the *end points* of the outline of the new district; they may or may not coincide with points which were common to the boundaries of two former districts, and, as a limiting case, may reduce to a single point.

The new external district will generally be *adjacent* to a certain number, say m, of the former border districts. In the nomenclature here adopted, it will have m variations and according as the end-points of its outline have been chosen it will completely cover up:

$m - 2$ districts
$m - 1$ "
m "

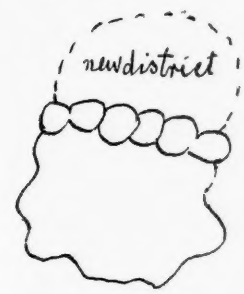

Fig. 1.

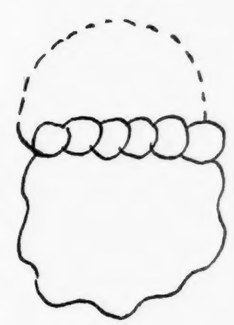

Fig. 2.

Fig. 3.

It is clear from this graphic process that the variations of every new district are found only among the *external* districts. *Internal* districts can receive no further variations.

Hence, bearing in mind the definitions of Art. 24, the external and internal districts in a map are respectively what we have called the elements of the *outer* and of the *inner* group, which terms have here an obvious concrete meaning.

Every map can obviously be drawn through super-addition of external districts. Hence from the remarks just made if we want the ideal elements hitherto dealt with, to become *symbols* for the districts of a map we must lay down the rule that whenever a polygeneous element (Ch. II, Art. 15) of the order m is added, $m - 2$ at least of its variation-elements pass on to the inner group (Ch. III, Art. 24).

In Fig. 3 the requirements of the theorem are fulfilled as all the variation districts of the new district have passed on to the inner group.

The inference to be drawn therefrom is the following:

Suppose it has been ascertained that the minimum of colors necessary to paint a certain particular map is k. The map can *a fortiori* be painted with $k + 1$ colors. Then without augmenting the number $k + 1$ of colors we can add any number of districts drawn as in Fig. 3, using only two of the colors for them.

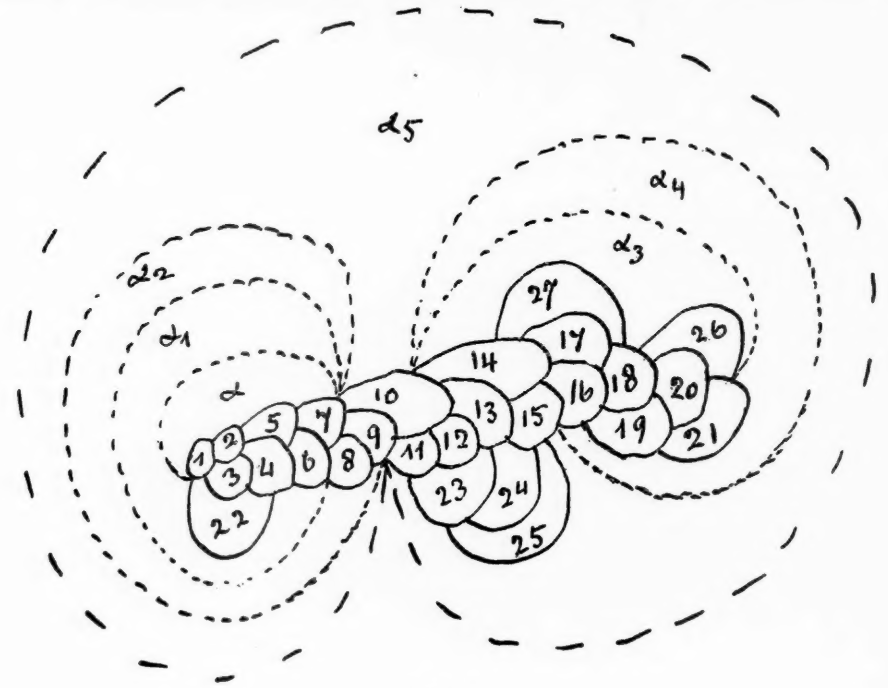

Fig. 4.

30.—*Example* 4.

In the map represented Fig. 4, the districts α, α_1, α_2, α_3, α_4, α_5 fulfil the requirements of the Theorem, Art. 28. Hence if the elements numbered 1, 2 .... 26, 27 can be annotated in three columns, it will be possible to annotate the whole map in four columns; i. e. to paint it with four colors.

As regards the partial aggregate of elements, 1, 2, ... 20, 21, we certainly have $k=3$. For 1, 2, 3, form a cycle of variations. The next element, 4, has, when introduced, only two variations, viz. 2 and 3; the next one, 5, when introduced, only two, viz. 2 and 4, etc. In fact, all the elements, 4, 5, ... 20, 21 are *duogeneous* (Ch. II, Art. 15), and we obtain at once the unique scheme of three columns here given, in which we have repeated the elements, so that each line may show the successive cycles of variations.

I	II	III
1	2	3
4	2	3
4	2	5
4	6	5
7	6	5
7	6	8
7	9	8
7	9	10
11	9	10

,

I	II	III
11	12	10
13	12	10
13	14	10
13	14	15
16	14	15
16	14	17
16	18	17
16	18	19
20	18	19
20	21	19

.

The remaining elements

22, 23, 24, 25, 26, 27

are all trigeneous, and follow the type of Fig. 1. Element 22 has, when added, the variations 1, 3, 4, shutting in element 3, which drops into the inner group; 23 has the variations 11, 12, 13, element 12 passing on to the inner group, etc. But the three variations of every one of these remaining elements always occupy two columns only of the scheme so far written, which circumstance enables us

to enter them successively into the same scheme of three columns, as here done, each bracket showing the three variations of the elements as and when added:

I	II	III.
1	22	3
4		
11	12	23
13		
13	24	23
		15
25	24	23
		15
20	18	26
	21	
27	18	17
	14	

We have then $k = 3$ for the elements numbered 1, 2, ... 26, 27; hence, the whole map can be annotated in four columns; in other words, be painted with four colors, *two* only being required for the α-districts.

31. Below the detail of the operations (comp. Arts. 23, 24).

Original Aggregate of Elements $\bar{\bar{E}} = 1, 2, \ldots 26, 27.$

We have here from the start an outer and an inner group, viz.

Outer group: $\bar{\bar{E}}' = 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 18, 19, 21, 22, 23, 25, 26, 27.$

Inner group: $\bar{\bar{E}}'' = 3, 12, 13, 17, 20, 24.$

A first partition must now be made in order to introduce the first additional element α; but as its variations can only be contained in the original outer group $\bar{\bar{E}}'$, we have here

First partition: $\bar{\bar{E}}' = \bar{\bar{A}} + \bar{\bar{B}},$

$\bar{\bar{A}} = 4, 6, 8, 9, 10, 11, 14, 15, 16, 18, 19, 21, 22, 23, 25, 26, 27;$

$\bar{\bar{B}} = 1, 2, 5, 7$ variation-elements of α.

Writing α in the scheme of three columns, we have symbolically

$$\left|\begin{array}{cccc} \alpha & & & \\ . & A & A & A \\ . & B & B & B \\ . & E' & E'' & E'' \end{array}\right|.$$

Observe that the elements of the set $\overline{E}''$ might be written in four columns. The above scheme is explicitly

$$\begin{array}{c|cccc|l} & \alpha & & & & \\ & & 4 & 6 & & \\ & & 11 & 9 & 8 & \\ & & & & 10 & \\ \overline{A} & & 16 & 14 & 15 & \text{outer group, } \overline{A}+\alpha \\ & & & 18 & & \\ & & 25 & 21 & 19 & \\ & & 27 & 22 & 23 & \\ & & & & 26 & \\ \hline \overline{B} & . & 1 & 2 & 5 & \\ & . & 7 & & & \text{inner group, } \overline{B}+\overline{E}'' \\ \overline{E}'' & & 13 & 12 & 3 & \\ & & 20 & 24 & 17 & \end{array}$$

First sub-partition:

$\overline{A}+\alpha=\overline{A}_1=\overline{B}_1$,

$\overline{A}_1=8, 9, 10, 11, 14, 15, 16\ 18, 19, 21, 23, 25, 26, 27$;

$\overline{B}_1=\alpha, 4, 6, 22$, variation-elements of α_1.

Representative scheme:

$$\left|\begin{array}{cccc} & & & \alpha_1 \\ A_1 & A_1 & A_1 & A_1 \\ B_1 & B_1 & B_1 & . \\ \alpha & & & \\ . & B & B & B \\ E' & E'' & E'' & E'' \end{array}\right|.$$

Specimen:

$$\begin{array}{c|cccc|}
 & & & & \alpha_1 \\
 & & 11 & 9 & 8 \\
 & & 16 & 14 & 10 \\
\overline{A}_1 & & 25 & 18 & 15 \\
 & & 27 & 21 & 19 \\
 & & & & 23 \\
 & & & & 26
\end{array}\Bigg\}\ \text{outer group, } \overline{A}_1+\alpha_1;$$

$$\begin{array}{c|cccc|}
\overline{B}_1 & \alpha & 4 & 6 & \cdot \\
 & & & 22 & \cdot \\
\overline{B} & \cdot & 1 & 2 & 5 \\
 & \cdot & 7 & & \\
\overline{E}'' & & 13 & 12 & 3 \\
 & & 20 & 24 & 17
\end{array}\Bigg\}\ \text{inner group, } \overline{B}_1+\overline{B}+\overline{E}''.$$

Second sub-partition:

$$\overline{A}_1+\alpha_1=\overline{A}_2+\overline{B}_2,$$

$\overline{A}_2 = 10,\ 11,\ 14,\ 15,\ 16,\ 18,\ 19,\ 21,\ 23,\ 25,\ 26,\ 27;$

$\overline{B}_2 = \alpha_1\ 8,\ 9,$ variation-elements of α_2.

Representative scheme:

$$\left|\begin{array}{cccc}
\alpha_2 & & & \\
A_2 & A_2 & A_2 & A_2 \\
\cdot & B_2 & B_2 & B_2 \\
 & & & \alpha_1 \\
B_1 & B_1 & B_1 & \cdot \\
\alpha & & & \\
\cdot & B & B & B \\
E'' & E' & E' & E'
\end{array}\right|.$$

Specimen:

	α_2	11	14	10	
		16	18	15	
$\overline{A}_2$		25	21	19	outer group, $\overline{A}_2 + \alpha_2$;
		27		23	
				26	
$\overline{B}_2$	$\cdot$		9	α_1	
	$\cdot$			8	
$\overline{B}_1$	α	4	6	$\cdot$	
			22	$\cdot$	inner group, $\overline{B}_2 + \overline{B}_1 + \overline{B} + \overline{E}''$.
$\overline{B}$	$\cdot$	1	2	5	
	$\cdot$	7			
$\overline{E}''$		13	12	3	
		20	24	17	

Third sub-partition:

$$\overline{A}_2 + \alpha_2 = \overline{A}_3 + \overline{B}_3,$$

$$\overline{A}_3 = \alpha_2\ 10,\ 11,\ 15,\ 16,\ 19,\ 21,\ 23,\ 25\,;$$

$$\overline{B}_3 = 14,\ 18,\ 26,\ 27,\ \text{variation-elements of } \alpha_3.$$

Representative scheme:

α_3			
A_3	A_3	A_3	A_3
α_2			
$\cdot$	B_3	B_3	B_3
$\cdot$	B_2	B_2	B_2
			α_1
B_1	B_1	B_1	$\cdot$
α			
$\cdot$	B	B	B
E''	E''	E''	E''

Specimen:

$$
\begin{array}{c|ccccl}
 & \alpha_3 & & & & \\
 & \alpha_2 & 11 & 21 & 10 & \\
\overline{A}_3 & & 16 & & 15 & \text{outer group, } \overline{A}_3+\alpha_3; \\
 & & 25 & & 19 & \\
 & & & & 23 & \\
\hline
\overline{B}_3 & \cdot & 27 & 14 & 26 & \\
 & \cdot & & 18 & & \\
\overline{B}_2 & \cdot & & 9 & \alpha_1 & \\
 & \cdot & & & 8 & \\
\overline{B}_1 & \alpha & 4 & 6 & \cdot & \text{inner group, } \overline{B}_3+\overline{B}_2+\overline{B}_1+\overline{B}+E' \\
 & & & 22 & \cdot & \\
\overline{B} & \cdot & 1 & 2 & 5 & \\
 & \cdot & 7 & & & \\
\overline{E}'' & & 13 & 12 & 3 & \\
 & & 20 & 24 & 17 &
\end{array}
$$

Fourth sub-partition:

$$\overline{A}_3+\alpha_3=\overline{A}_4+\overline{B}_4,$$
$$\overline{A}_4=\alpha_2\ 10,\ 11,\ 15,\ 23,\ 25;$$
$$\overline{B}_4=\alpha_3\ 16,\ 19,\ 21,\ \text{variation-elements of } \alpha_4.$$

Representative scheme:

$$
\begin{array}{|cccc|}
\hline
 & & & \alpha_4 \\
A_4 & A_4 & A_4 & A_4 \\
 & & & \alpha_2 \\
B_4 & B_4 & B_4 & \cdot \\
\alpha_3 & & & \\
\cdot & B_3 & B_3 & B_3 \\
B_2 & B_2 & B_2 & \cdot \\
\alpha_1 & & & \\
\cdot & B_1 & B_1 & B_1 \\
 & & & \alpha \\
B & B & B & \cdot \\
E'' & E'' & E'' & E''
\end{array}
$$

Specimen:

$$\begin{array}{c|cccc|l}
 & & & & \alpha_4 & \\
 & & 11 & & 10 & \\
\overline{A}_4 & & 25 & & \alpha_2 & \text{outer group, } \overline{A}_4+\alpha_4; \\
 & & & & 15 & \\
 & & & & 23 & \\
\hline
\overline{B}_4 & 19 & 16 & 21 & \cdot & \\
 & \alpha_3 & & & & \\
\overline{B}_3 & \cdot & 27 & 14 & & \\
 & \cdot & & 18 & 26 & \\
\overline{B}_2 & 8 & & 9 & \cdot & \\
 & \alpha_1 & & & \cdot & \text{inner group, } \overline{B}_4+\overline{B}_3+\overline{B}_2+\overline{B}_1+\overline{B}+\overline{E}''. \\
\overline{B}_1 & \cdot & 4 & 6 & \alpha & \\
 & \cdot & & 22 & & \\
\overline{B} & 5 & 1 & 2 & \cdot & \\
 & & 7 & & \cdot & \\
\overline{E}'' & & 13 & 12 & 3 & \\
 & & 20 & 24 & 17 &
\end{array}$$

Fifth sub-partition:

$$\overline{A}_4+\alpha_4=\overline{A}_5+\overline{B}_5,$$
$$\overline{A}_5=0,$$
$$\overline{B}_5=11,\ 23,\ 25,\ 15,\ 10,\ \alpha_2,\ \alpha_4,\ \text{variation-elements of } \alpha_5.$$

Representative scheme:

$$\left|\begin{array}{cccc}
\alpha_5 & & & \\
\cdot & B_5 & B_5 & B_5 \\
 & & & \alpha_4 \\
 & & & \alpha_2 \\
B_4 & B_4 & B_4 & \cdot \\
\alpha_3 & & & \\
\cdot & B_3 & B_3 & B_3 \\
B_2 & B_2 & B_2 & \cdot \\
\alpha_1 & & & \\
\cdot & B_1 & B_1 & B_1 \\
 & & & \alpha \\
B & B & B & \cdot \\
E'' & E'' & E'' & E''
\end{array}\right|.$$

Specimen:

	α_5				outer group, α_5;
$\overline{B}_5$	•	11		10	
	•	25		15	
	•			23	
	•			α_4	
	•			α_2	inner group, $\overline{B}_5+\overline{B}_4+\overline{B}_3+\overline{B}_2+\overline{B}_1+\overline{B}+\overline{E}''$.
$\overline{B}_4$	19	16	21	•	
	α_r			•	
	etc., identical with the precedent.				

The outer group reducing to the single element, α_5, the process of sub-partitioning is closed, and the whole map stands annotated in four columns, the α-elements in two.

32. If we add two new elements, e_1, e_2, forming a cycle of variations with α_5, the unique external element of our map (Fig. 5), we find ourselves pre-

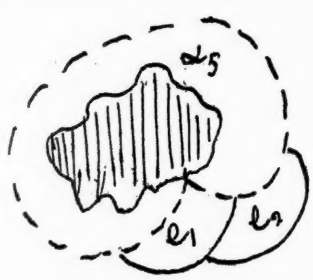

Fig. 5.

cisely in the same situation as in the original map, when we started with the cycle of variations given by the elements 1, 2, 3. We could designate the three elements, e_1, e_2, α_5, in their turn by the numbers 1, 2, 3, and on these elements build up the same map as before, or another, first by adding any quantity of duogeneous elements, thus leaving the minimum of columns $k=3$ unaltered; next adding, as we did before, polygeneous elements *so chosen as to preserve the same minimum* 3, and, finally, a new set of α-elements fulfilling the requirements of the theorem, i. e. symbolizing districts drawn as in Fig. 3.

In annotating this second map, the elements of the inner group in the first map may be entirely disregarded, and the new map annotated separately without reference to the first. When completed, the two final schemes representing the first and the second map, can be superposed, they having only one element in common, viz. α_5; they then combine into one scheme descriptive of the aggregate of the two maps.

We have supposed, for clearness, all the duogeneous elements to be added first, and next all the polygeneous elements which leave the minimum of columns unaltered, but the two operations can be alternated at will.

If, instead of making $\overline{A}_5 = 0$ in the fifth sub-partition, we had assigned two elements to that set, e. g.

$$\overline{A}_5 = 11,\ 23,$$
$$\overline{B}_5 = \alpha_2,\ \alpha_4,\ 10,\ 15,\ 25,$$

the outer group would have been α_5 11 23 and the final scheme

$$\begin{array}{c|ccc|l}
\overline{A}_5 & \alpha_5 & 11 & 23 & \text{outer group,} \\
\hline
 & \cdot & 25 & 10 & \\
 & \cdot & & 15 & \\
\overline{B}_5 & \cdot & & \alpha_2 & \text{inner group.} \\
 & & & \alpha_4 & \\
 & \text{etc.} & & &
\end{array}$$

The elements 11, 23 are connected by a variation, and both with α_5 by a permanence (Fig. 6), but as in the final scheme, these three elements occur in

Fig. 6.

three different columns, we can consider them fictitiously as forming a cycle of variation, and build up a new map on them just as before. When completed, the two final schemes will be superposable as the only three elements they have in common, viz. α_5, 11, 23, can be written in both in the same three columns.

A glance at the map (Fig. 6) shows that our fictitious hypothesis is tantamount to tracing the last α-district, viz. α_5 after the type of Fig. 1 instead of Fig. 3, for then the three contour-districts, α_5, 11, 23, will actually form a cycle of variations. In this case, the last α-element does not fulfill the requirements of the theorem, yet no increase in the number of colors is necessary.

Superconstructions of maps in the two cases just recorded can evidently be repeated indefinitely, but if the last α-element left more than two of the original elements on the border, the argument would fail, as on account of the sub-partitions in the second map one cannot assert *a priori* that the elements common to both maps can be made to occupy the same columns in both final schemes.

33. The illustrations here given of the theorem of Art. 28, shows that a great variety of maps can be painted with four colors in consequence of Proposition I alone. That this proposition cannot of itself cover the whole ground is evident, and it is hardly necessary to observe that the example has not been introduced as leading to a solution of the geographical problem, but merely in order to illustrate how general results in the annotation of ideal elements can be applied to maps.

The problem in the annotation of ideal elements to which the general geographical problem corresponds, depends on Proposition II.